Inverse Scattering in Two Dimensions: Scattering Theory and Asymptotics

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References

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Overview

In this lecture, we'll consider nonlinear scattering and asymptotics for two completely integrable dispersive PDE's:

(1) The DS II equation, considered in Lecture 3:

$$\begin{aligned} iq_t + 2(\partial_z^2 + \partial_{\overline{z}}^2)q + (g + \overline{g})q &= 0 \\ \partial_{\overline{z}} g + 4\varepsilon \partial_z (|q|^2) &= 0 \\ q(0,x,y) &= q_0(x,y) \end{aligned}$$

where $\varepsilon = 1$ (resp. $\varepsilon = -1$) for the defocussing (resp. focussing) equation

(2) The Kadomtsev-Petviashvili equation,

$$\begin{aligned} (u_t + u_{xxx} + 6uu_x)_x &= 3\lambda u_{yy} \\ u(0,x,y) &= u_0(x,y) \end{aligned}$$

where $\lambda = 1$ (resp. $\lambda = -1$) is the KPI (resp. KP II) equation.

We will discuss *nonlinear scattering* (the existence of scattering asymptotics for the solutions of these PDE's) and *long-time asymptotics*

DS II Asymptotics: Pointwise Behavior

Theorem *(Perry) Suppose that* $q_0 \in H^{1,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *and* $(\mathcal{S}q_0)(0) = 0$. *Then*

$$q(t,z) = v(t,z) + o(t^{-1})$$

where $v(t,z)$ *solves the problem*

$$\begin{cases} i\partial_t v + 2(\partial_z^2 + \partial_{\overline{z}}^2)v = 0, \\ \qquad\qquad v(0,z) = \mathcal{F}_a\left(\mathcal{S}(q_0)\right) \end{cases}$$

The proof depends on a careful study of the $\overline{\partial}$ problem

$$\begin{aligned} \partial_{\overline{k}}\nu_1 &= \frac{1}{2} e^{-itS(z,k,t)}\overline{r_0 \nu_2} \\ \partial_{\overline{k}}\nu_2 &= \frac{1}{2} e^{itS(z,k,t)}\overline{r_0 \nu_1} \\ \lim_{|k|\to\infty}(\nu_1,\nu_2) &= (1,0) \end{aligned}$$

where $r_0 = \mathcal{S}(q_0)$ and $S(z,k,t) = (kz - \overline{k}\overline{z})/it + 4\,\mathrm{Re}(k^2)$.

DS II Asymptotics: Pointwise Behavior

If $q_0 \in H^{1,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, then $r_0 \in H^{1,1}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$. The reconstruction formula is

$$q(t,z) = \frac{1}{\pi} \int_{\mathbb{C}} e^{itS(z,k,t)} r_0(k) \nu_1(z,k,t)\, dA(k), \quad S(z,k,t) = (kz - \overline{k}\overline{z})/it + 4\,\mathrm{Re}(k^2)$$

and

$$\begin{aligned} \partial_{\overline{k}} \nu_1 &= \frac{1}{2} e^{-itS(z,k,t)} \overline{r_0 \nu_2} \\ \partial_{\overline{k}} \nu_2 &= \frac{1}{2} e^{itS(z,k,t)} \overline{r_0 \nu_1} \\ \lim_{|k| \to \infty} (\nu_1, \nu_2) &= (1, 0) \end{aligned}$$

and $\nu(t,z)$ is exactly

$$\nu(t,z) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k)\, dA(k)$$

so the key issue is large-t asymptotic behavior of $\nu_1(z,k,t)$.

DS II Asymtotics: Pointwise Behavior

$$q(t,z) - v(t,z) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k) \left(v_1(z,k,t) - 1 \right) \, dA(k).$$

To study $v_1 - 1$, return to the integral equation for v_1. Let

$$M\psi = \frac{1}{2} P_k \left(e^{-itS(z,\cdot,t)} \overline{r_0 \psi} \right), \qquad (P_k f)(k) := \frac{1}{\pi} \int \frac{f(\zeta)}{k - \zeta} \, dA(\zeta)$$

For $r_0 \in H^{1,1}(\mathbb{R}^2)$, the operator M is a compact operator on $L^p_k(\mathbb{R}^2)$ for any $p \in (2, \infty)$, and $(I - M^2)^{-1}$ is bounded on $L^p(\mathbb{R}^2)$. We have

$$v_1 = 1 + (I - M^2)^{-1} M^2 1$$

so the crucial estimates are on time-decay of $M^2 1$. The phase function

$$S(z,k,t) = 4 \, \mathrm{Re}((k - k_c)^2) + S_0, \quad S_0 = \frac{1}{4} \mathrm{Re}(z^2/t^2)$$

has a single critical point at $k_c = iz/4t$.

DS II Asymptotics: Pointwise Behavior

$$M\psi = \frac{1}{2}P_k\left(e^{-itS(z,\cdot,t)}\overline{r_0\psi}\right), \quad S(z,k,t) = 4\,\mathrm{Re}(k-k_c)^2 + S_0$$

$$\nu_1 - 1 = (I - M^2)^{-1}M^2 1$$

Let

$$\gamma = \|r_0\|_{H^{1,1}} + \|r_0\|_{C^0}$$

Using stationary phase methods, we show that for any $p > 2$ and a suitable cutoff function χ with support in an $\mathcal{O}\left(t^{-\frac{1}{4}}\right)$ neighborhood of k_0:

$$\|M(\chi 1)\|_p \lesssim_p \gamma t^{-1/4-1/(2p)} \qquad \|M((1-\chi)1)\|_p \lesssim_p \gamma t^{-3/4}$$

$$\left\|M^4 1\right\|_p \lesssim_p \gamma^4 t^{-1-1/(2p)}$$

$$\|M\|_{L^p \to L^p} \lesssim_p \gamma \qquad \|M\chi\|_{L^p \to L^p} \lesssim_p \gamma t^{-1/4}$$

$$\left\|M^2\right\|_{L^p \to L^p} \lesssim_p t^{-\frac{1}{4}}\gamma^2$$

DS II Asymptotics: Pointwise Behavior

$$q(t,z) - v(t,z) = \frac{1}{\pi} \int e^{itS(z,k,t)} r_0(k) \left(\nu_1(z,k,t) - 1\right) \, dA(k)$$

$$\nu_1 - 1 = (I - M^2)^{-1} M^2 1$$

Using the operator norm estimates on M and the estimates on $M^4 1$ we can show that for $r_0 \in H^{1,1}(\mathbb{R}^2) \cap C^0(\mathbb{R}^2)$,

$$\sup_{z \in \mathbb{C}} \left| u(z,t) - v(z,t) - \frac{1}{\pi} \int e^{itS} r_0 (M^2 1) \, dk \right| \lesssim_{p,\gamma} t^{-1-1/p}$$

The $o(t^{-1})$ estimate on the remaining term is obtained by a stationary phase analysis of the four integrals

$$I_1 = \int e^{itS} r_0 \, M \left[(1-\chi) M(\chi) \right] \, dA(k)$$

$$I_2 = \int e^{itS} r_0 \, M \left[\chi M(1-\chi) \right] \, dA(k)$$

$$I_3 = \int e^{itS} r_0 \, M \left[(1-\chi) M(1-\chi) \right] \, dA(k)$$

$$I_4 = \int e^{itS} r_0 \, M \left[\chi M(\chi) \right] \, dA(k)$$

DS II Asymptotics: Pointwise Behavior

Estimates on defocussing DS II equations were first obtained by Kiselev (1997). On the one hand, Kiselev's assumptions were more restrictive than ours, and on the other hand he considers both the defocussing case and the focussing case, making a small data assumption. Much of the analysis here is inspired by his work.

It is interesting to note that the DS II asymptotics show *no* logarithmic phase shift. Instead, they directly mirror the asymptotic behavior of solutions to the linear problem

$$i\partial_t v + 2(\partial_z^2 + \partial_{\bar{z}}^2)q = 0$$

DS II: Plancherel Theorem for $\mathcal{S}$

Nachman, Regev and Tataru (2020) proved a pathbreaking result on the DS II equation including nonlinear scattering. All are consequences of the following fundamental theorem on the scattering transform.

Theorem *(Nachman, Regev, Tataru) The nonlinear scattering transform* $\mathcal{S} : q \mapsto \mathbf{s}$ *is a* C^1 *diffeomorphism* $\mathcal{S} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ *satisfying:*

(i) $\|\mathcal{S}q\|_{L^2} = \|q\|_{L^2}$

(ii) $|(\mathcal{S}q)(k)| \lesssim C(\|q\|_{L^2})(\mathcal{M}\widehat{q})(k)$ *for a.e.* k

(iii) $q \to \mathcal{S}q$ *is uniformly bi-Lipschitz*

(iv) $\left\| \dfrac{\delta \mathcal{S}}{\delta q} \right\|_{L^2 \to L^2} \lesssim C(\|q\|_{L^2})$

(v) $\mathcal{S}^{-1} = \mathcal{S}$

(vi) $\mathcal{S}$ *is a symplectomorphism, i.e.,* $\omega_1 \left(\left.\dfrac{\delta S}{\delta q}\right|_q q_1, \left.\dfrac{\delta S}{\delta q}(q_2)\right|_q q_2 \right) = \omega_2(q_1, q_2)$

where

$$\omega_1(q_1, q_2) = -\operatorname{Im} \int q_1 \overline{q_2} \, dz, \qquad \omega_2(t_1, t_2) = -\operatorname{Im} \int t_1 \overline{t_2} \, dk$$

DS II: Well-Posedness and Scattering

Let $U(t)$ be the solution operator for the linear DS II equation. As a consequence of their theorem and the solution formula $q(t) = \mathcal{S}^{-1}\left(e^{4it\,\mathrm{Re}(k^2)}\mathcal{S}q_0\right)$, Nachman, Regev, and Tataru proved:

Theorem

The defocussing DS II equation is globally well-posed on $L^2(\mathbb{R}^2)$ and solutions scatter. More precisely, the solution satisfies

(i) $q(t,z) \in C(\mathbb{R}, L^2_z(\mathbb{C})) \cap L^4_{t,z}(\mathbb{R} \times \mathbb{C})$

(ii) $\|q(t)\|_{L^2} = \|q(0)\|_{L^2}$ *for all* $t \in \mathbb{R}$ *and* $\|q\|_{L^4_{t,z}} \lesssim C(\|q_0\|_{L^2})$

(iii) $|q(t,z)| \leq C(\|q_0\|_{L^2})\mathcal{M}q^{\mathrm{lin}}(t,z)$ *where* q^{lin} *solves the linear DS II equation with initial data* $\mathcal{S}q_0$

(iv) $\|q_1(t,\cdot) - q_2(t,\cdot)\|_{L^2} \leq C(R)\,\|q_1(0,\cdot) - q_2(0,\cdot)\|_{L^2}$ *provided* $\|q_1(0,\cdot)\|_{L^2}, \|q_2(\cdot,0)\|_{L^2} \leq R$.

(v) *For* $q_0 \in L^2(\mathbb{C})$ *there are* $q_-, q_+ \in L^2(\mathbb{C})$ *so that*

$$\lim_{t\to-\infty} \|q(t,\cdot) - U(t)q_-(\,\cdot\,)\|_{L^2} = \lim_{t\to+\infty} \|q(t,\cdot) - U(t)q_+(\,\cdot\,)\|_{L^2} = 0$$

Ideas of the Proof - Fractional Integrals

Let

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

be the Hardy-Littlewood maximal function for functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and recall $\mathcal{M} : L^p \to L^p$ for $1 < p \leq \infty$. (think $\alpha = 1, n = 2$

Theorem *For* $0 < \alpha < n$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$:

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \leq c_{n,\alpha} \left(\lambda^{n-\alpha} M\widehat{f}(0) + \lambda^{-\alpha} Mf(x)\right)$$

$$\left|(-\Delta)^{-\frac{\alpha}{2}} f(x)\right| \leq c_{n,\alpha} \left(\mathcal{M}\widehat{f}(0)\right)^{\frac{\alpha}{n}} \left(\mathcal{M}f(x)\right)^{1-\frac{\alpha}{n}}$$

Corollary *For* $q \in L^2(\mathbb{C})$:

(a) $\left|\overline{\partial}^{-1}(e_{-k}q)(x)\right| \lesssim (\mathcal{M}\widehat{q}(k))^{\frac{1}{2}} (\mathcal{M}q(x))^{\frac{1}{2}}$

(b) $\left\|\overline{\partial}^{-1}(e_{-k}q)\right\|_{L^4} \lesssim \|q\|_{L^2}^{\frac{1}{2}} (M\widehat{q}(k))^{\frac{1}{2}}$

Ideas of the Proof - ΨDO Estimates

Theorem *Let* $0 \leq \alpha < n$ *and suppose* $a(x,\xi)$ *satisfies*

(i) $\int_{\mathbb{R}^n \times \mathbb{R}^n} |a(x,\xi)|^{\frac{2n}{n-\alpha}} \, d\xi \, dx < \infty$ *and*

(ii) $\left\| (-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi) \right\|_{L_\xi^{\frac{2n}{n+\alpha}}} \in L_x^{\frac{2n}{n-\alpha}}$

Then, the pseudodifferential operator

$$a(x,D)f(x) := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} a(x,\xi) \widehat{f}(\xi) \, d\xi$$

is bounded on L^2 *with*

$$\|a(x,D)\|_{L^2 \to L^2} \leq \left\| (-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\xi) \right\|_{L_x^{\frac{2n}{n-\alpha}} L_\xi^{\frac{2n}{n-\alpha}}}$$

and

$$|a(x,D)f(x)| \leq c_{\alpha,n} (\mathcal{M}f(x))^{\frac{\alpha}{n}} \left\| (-\Delta_\xi)^{\frac{\alpha}{2}} a(x,\cdot) \right\|_{L^{\frac{2n}{n+\alpha}}} \|f\|_{L^2}.$$

Ideas of the Proof: Scattering Solutions

To analyze the direct scattering transform

$$\mathcal{S}q(k) = -\frac{i}{\pi}\int_{\mathbb{R}^2} e_k(z)\overline{q(z)}m^1(z,k)\,dz$$

we need fine estimates on m^1. Recall that

$$m^1 = \frac{1}{2}(m_+ + m_-)$$
$$m^2 = e_{-k}(\overline{m_+ - m_-})$$

where

$$\frac{\partial}{\partial \overline{z}}(m^\pm - 1) = e_{-k}(\overline{m_\pm} - 1) \pm e_{-k}q$$

This naturally focuses attention on the inhomogeneous problem

$$\overline{\partial}u + e_{-k}q\overline{u} = e_{-k}f$$

or equivalently the operator L_q^{-1} where

$$L_q u = \overline{\partial}u + q\overline{u}$$

Ideas of the Proof: Concentration Compactness

Recall the homogeneous Sobolev spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ and the dual space $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^2) \supset L^{\frac{4}{3}}(\mathbb{R}^2)$. If

$$L_q u = \overline{\partial} u + q\overline{u},$$

then

$$L_q^{-1} f = (I + \overline{\partial}^{-1}(q\,\cdot))^{-1} \overline{\partial}^{-1} f$$

The goal is to show that

$$\left\| L_q^{-1} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2}).$$

Step 1: For $q \in L^2(\mathbb{R}^2)$, $L_q : \dot{H}^{\frac{1}{2}} \to \dot{H}^{-\frac{1}{2}}$ is invertible with

$$\left\| L_q^{-1} f \right\|_{\dot{H}^{\frac{1}{2}}} \leq C(q) \left\| f \right\|_{\dot{H}^{-\frac{1}{2}}}$$

Step 2: $q \to L_q^{-1}$ is a smooth map and $q \to C(q)$ is locally Lipschitz

Ideas of the Proof: Concentration Compactness

Recall $L_q u = \overline{\partial} u + q\overline{u}$, The goal is to show that

$$\left\| L_q^{-1} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \leq C(\|q\|_{L^2}).$$

Step 3: If

$$C(R) = \sup\{C(q) : \|q\|_{L^2} \leq R\},$$

then $C(R) < \infty$ for R small and $C(R)$ is nondecreasing and continuous.

Step 4: Frame a proof-by-contradiction argument to show that $C(R)$ is finite for all R.

If not, choose R_0 minimal so that $C(R_0) = \infty$. By continuity there is a sequence $\{q_n\}$ with $\|q_n\|_{L^2} \nearrow R_0$ so that

$$\left\| L_{q_n}^{-1} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \to \infty$$

If a subsequence of $\{q_n\}$ converged in L^2 we could obtain a contradiction by

$$\left\| L_{q_n} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}}^{-1} \to \left\| L_q^{-1} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} \neq \infty$$

Ideas of the Proof: Concentration Compactness

Symmetry under translation and scaling is an obstruction to compactness: If $S(\lambda, y)q(x) = \lambda q(\lambda(x-y))$, then

$$C(q) = C(S(\lambda, y)q).$$

So the best we can expect is compactness modulo the action of these symmetries

Step 5: Expand the pertubative theory to $\dot{B}_{\infty}^{-\frac{1}{3},3}(\mathbb{R}^2) \supset L^2(\mathbb{R}^2)$ (Sobolev embedding) – note the bilinear estimate

$$\|qu\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|q\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}} \|u\|_{\dot{H}^{\frac{1}{2}}}$$

Ideas of the Proof: Concentration Compactness

Step 6: Prove that any bounded subsequence $\{q_n\}$ in L^2 has a subsequence admitting the following "profile decomposition" for any $l \in \mathbb{N}$:

$$q_n = \sum_{k=1}^{l} S(\lambda_n^k, y_n^k) q^k + q_n^l$$

where the functions q^j (profiles) are all in L^2 for all $j \in \mathbb{N}$, and the remainders q_n^l are uniformly bounded in L^2 and satisfy

$$\lim_{l\to\infty} \limsup_{n\to\infty} \left\| q_n^l \right\|_{\dot{B}_{\infty}^{-\frac{1}{3},3}} = 0$$

and (λ_k^n, y_k^n) is a sequence in $\mathbb{R}^+ \times \mathbb{R}^2$ with separation of profiles:

$$\lim_{n\to\infty} \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} = \infty \quad \text{or} \quad \lambda_n^j = \lambda_n^k, \lim_{n\to\infty} |y_n^j - y_n^k| \lambda_n^j = 0$$

(cf. Gérard 1998). This decomposition allows a perturbative analysis of L_{q_n} to show that $\lim_{n\to\infty} \left\| L_{q_n}^{-1} \right\|_{\dot{H}^{-\frac{1}{2}} \to \dot{H}^{\frac{1}{2}}} < \infty$

The KP Equation

The KP equation

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = -3\sigma^2 \partial_x^{-1} \partial_y^2 u$$

is a completely integrable, dispersive PDE. The case $\sigma = i$ is the KP I equation, and the case $\sigma = 1$ is the KP II equation.

The KP equation is the compatibility condition for the system

$$(\sigma\partial_y + \partial_x^2 + u)\psi = 0, \quad (1)$$

$$(\partial_t + 4\partial_x^3 + 6u\partial_x + 3(u_x - \sigma\partial_x^{-1}\partial_y u))\psi = 0. \quad (2)$$

(Dryuma 1974), or, in the Lax representation

$$L = \sigma\partial_y + \partial_{xx}$$

$$A = 4\partial_x^3 + 6u\partial_x + 3(u_x - \sigma\partial_x^{-1}\partial_y u)$$

where $\dot{L} = [A, L]$ gives the KP equation. As usual, equation (1) defines a spectral problem and scattering data, while equation (2) determines the time-evolution of scattering data under the KP flow.

The KP Equation: Inverse Scattering

$$\partial_t u + \partial_x^3 u + 6u\partial_x u = -3\sigma^2 \partial_x^{-1}\partial_y^2 u$$

$$(\sigma\partial_y + \partial_x^2 + u)\psi = 0,$$

$$(\partial_t + 4\partial_x^3 + 6u\partial_x + 3(u_x - \sigma\partial_x^{-1}\partial_y u))\psi = 0.$$

For KP I ($\sigma = i$):

- The spectral problem is a time-dependent Schrödinger equation
- The scattering data are transmission coefficients $T^{\pm}(k,l)$ with discontinuity at $k = l$
- The inverse problem is a nonlocal Riemann-Hilbert problem

For KP II ($\sigma = 1$):

- The spectral problem is a heat equation
- The scattering data is a function $F : \mathbb{C} \to \mathbb{C}$ with discontinuity at $\operatorname{Re} k = 0$
- The inverse problem is a $\bar{\partial}$ problem

Well-Posedness Results

It is known (Molinet, Saut, Tzvetkov, *Math. Ann.* 2004) that for initial data in the space Z with norm

$$\|u\|_Z = \|u\|_{L^2} + \|u_{xxx}\|_{L^2} + \|u_y\|_{L^2} + \|u_{xy}\|_{L^2} + \left\|\partial_x^{-1} u_y\right\|_{L^2} + \left\|\partial_x^{-2} u_{yy}\right\|_{L^2}$$

there exists a unique global solution with $u, u_y, u_{xx} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and with conserved mass and energy:

$$M(u) = \int_{\mathbb{R}^2} |u|^2, \qquad E(u) = \frac{1}{2}\int_{\mathbb{R}^2} u_x^2 + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_x^{-1} u_y)^2 - \frac{1}{6}\int_{\mathbb{R}^2} u^3.$$

Large-Time Asymptotics: Linear KP Equation

We seek to determine large-time asymptotics of solutions to the KP I equation with small data. As a "warm-up" we consider Cauchy problem for the linear KP equation

$$\begin{cases} \partial_t v + \partial_x^3 v = 3\lambda \partial_x^{-1} \partial_y^2 u \\ v(0,x,y) = v_0(x,y) \end{cases}$$

where $\lambda = 1$ for KP I, $\lambda = -1$ for KP II. This equation has a solution via Fourier analysis:

$$v(t,x,y) = \frac{1}{2\pi} \int e^{it(p\xi + q\eta + (p^3 - 3\lambda p^{-1} q^2))} \widehat{v}_0(p,q)\, dp\, dq$$

where $\xi = x/t$, $\eta = y/t$.

Large-Time Asymptotics: Linear KP Equations

Recall

$$v(t,x,y) = \frac{1}{2\pi} \int e^{it(p\xi + q\eta + (p^3 - 3\lambda p^{-1} q^2))} \widehat{v_0}(p,q)\, dp\, dq$$

where $\xi = x/t, \eta = y/t$.

For the linear KP I equation, if we set $p = l - k$, $q = k^2 - l^2$, we conclude

$$v(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;\xi,\eta)} \widehat{v_0}(l-k, -(l^2-k^2))\, |l-k|\, dk\, dl$$

where

$$S(k,l;\xi,\eta) = (l-k)\xi - (l^2-k^2)\eta + 4(l^3-k^3).$$

The phase function S has four nondegenerate critical points at

$$(l,k) = \left(\frac{\eta}{12} \pm \frac{\sqrt{\eta^2 - 12\xi}}{12}, \frac{\eta}{12} \pm \frac{\sqrt{\eta^2 - 12\xi}}{12}, \right)$$

provided

$$\eta^2 - 12\xi > 0$$

Large-Time Asymptotics: Linear KP Equations

Recall

$$v(t,x,y) = \frac{1}{2\pi}\int e^{it(p\xi + q\eta + (p^3 - 3\lambda p^{-1}q^2))}\widehat{v_0}(p,q)\,dp\,dq$$

where $\xi = x/t$, $\eta = y/t$.

For the linear KP II equation, if we set $p = -(k+\overline{k})$, $q = i(k^2 - \overline{k}^2)$, then

$$v(t,x,y) = \frac{1}{\pi}\int e^{itS(k,\overline{k};\xi,\eta)} i(k+\overline{k})\widehat{v_0}(-(k+\overline{k}, i(k^2-\overline{k}^2))\,dk \wedge d\overline{k}$$

where

$$S(k,\overline{k};\xi,\eta) = -(k+\overline{k})\xi - i(k^2 - \overline{k}^2)\eta + 4(k^3 + \overline{k}^3)$$

The phase function has nondegenerate critical points at

$$(k,\overline{k}) = \left(\frac{i\eta}{12} \pm \frac{\sqrt{-(\eta^2+12\xi)}}{12}, -\frac{i\eta}{12} \pm \frac{\sqrt{-(\eta^2+12\xi)}}{12}\right)$$

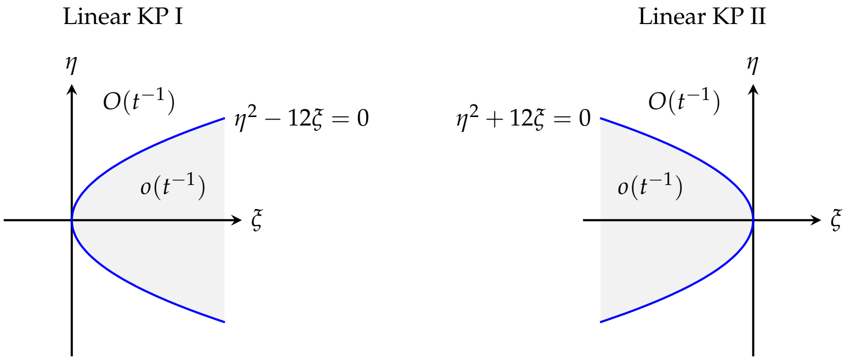
provided

$$\eta^2 + 12\xi < 0$$

Linear KP Equation: Asymptotic Behavior

For the linear KP equations, we expect the following pointwise asymptotics:

- Scattering ($O(t^{-1})$ behavior) if $\eta^2 - 12\xi > 0$ (Linear KP I)
- Scattering ($O(t^{-1})$ behavior) if $\eta^2 + 12\xi < 0$ (Linear KP II)



These asymptotics show what to expect for the nonlinear equations.

Asymptotics: Previous Work by IST Methods

Kiselev (2004): large-time asymptotics for KP II for small initial data

(i) For $-(\eta^2 + 12\xi)t^{\frac{1}{3}} \gg 1$,

$$u(t,x,y) \underset{t\to\infty}{\sim} -4t^{-1}\frac{\pi}{12ir} f\left(\frac{1}{2}r + \frac{i\eta}{12}\right) e^{-11itr} + \text{c.c.} + o(t^{-1})$$

(ii) For $(\eta^2 + 12\xi)t^{\frac{1}{3}} \gg 1$,

$$u(t,x,y) = o(t^{-1})$$

(iii) For $|12\xi + \eta^2| \ll 1$,

$$u(t,x,y) \underset{t\to\infty}{\sim} 8it^{-1} f(i\eta/12)\sqrt{\pi}\left(\int_0^\infty \sqrt{p_1}\cos(8p_1^3 - zp_1)\,dp_1 + \int_0^\infty \sqrt{p_1}\sin(8p_1^3 - zp_1)\,dp_1\right)$$

Here f is the scattering data, $r = \sqrt{-(\eta^2+12\xi)}$ and $z = 8t^{\frac{2}{3}}\left(\frac{\eta^2}{12} + \xi\right)$

Asymptotics: Previous Work by IST Methods

Manakov, Santini, and Takhtajan (1980) computed large-time asymptotics for KP I with small initial data in the region $\eta^2 - 12\xi > 0$ (scattering region) using a stationary phase arguments. They obtained

$$u(t,x,y) \underset{t\to\pm\infty}{\sim} -\frac{2}{t}\varphi_\xi(\xi,\eta)\left[K^{\pm}_{-1,+1}(\xi,\xi,\eta)e^{it\varphi(\xi,\eta)} + \text{c.c.}\right]$$

where

$$\varphi(\xi,\eta) = \frac{1}{108}\left(\eta^2 - 12\xi\right)^{\frac{3}{2}}$$

and $K^{\pm}_{-1,+1}$ is obtained from the solution of the Gelfand-Levitan-Marchenko equation

$$K(x,x',y,t) + F(x,x',y,t) + \int_{-\infty}^{x} K(x,x'',y,t)F(x'',x,y,t)\,dx'' = 0$$

where, for scattering data $f(k,k')$,

$$F(x,x',y,t) = \frac{1}{2\pi}\iint f(k,k')e^{i(kx-k'x'-(k^2-k'^2)y+4(k^3-k'^3)t)}f(k,k')\,dk\,dk'$$

Take-Aways

From these results, we see:

(1) There are three space-time regions where the KP equations have different asymptotics: a scattering region, a transition region, and a no-scattering region

(2) These regions are exactly the regions where the phase function S for the linear problem has nondegenerate critical points, (almost) degenerate critical points, and no critical points

A close examination of the proofs shows that it is critically important to understand the scattering solutions as determined by the inverse problem: the respective reconstruction formulas are

$$u(t,x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\left(\int_{\mathbb{R}^2} e^{itS(k,l;\xi,\eta)} f(k,l)\mu(l,x;y,t)\,dk\,dl\right) \qquad \text{KP I}$$

$$u(t,x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\left(\int_{\mathbb{C}} e^{itS(k;\xi,\eta)} F(k)\varphi(k,x,y,t)dk \wedge d\bar{k}\right) \qquad \text{KP II}$$

where μ satisfies a nonlocal Riemann-Hilbert problem and φ satisfies a $\bar{\partial}$ problem, each with time-evolved scattering data.

Long-Time Asymptotics for the KP I Equation

Denote

$$\widetilde{u}(l;y) = \frac{1}{\sqrt{2\pi}} \int e^{-ilx} u(x,y)\, dy$$

and

$$\|\widetilde{u}\|_{L^{2,-1}} = \left(\int_{\mathbb{R}^2} |l|^{-1} |\widetilde{u}(l;y)|^2 \, dl\, dy \right)^{\frac{1}{2}}$$

Donmazov, Liu, and Perry proved:

Theorem *Suppose that $u_0 \in Z_w$ with $\|\widetilde{u}_0\|_{L^1} < \sqrt{2\pi}$ and $\|\widetilde{u}_0\|_{L^{2,-1}}$ small. Let $u(t,x,y)$ be the solution to KPI with initial data u_0, and let $a = 12\xi - \eta^2$. The following estimates hold:*

(i) *For $a > c > 0$ (no scattering region),*

$$u(t,x,y) = o(t^{-1}).$$

(ii) *For $a < -c < 0$ (scattering region),*

$$u(t,x,y) = \mathcal{O}\left(t^{-1}\right).$$

(iii) *For $|a| < c$ (transition region),*

$$u(t,x,y) = \mathcal{O}\left(t^{-2/3}\right).$$

Ideas of the Proof

The reconstruction formula is

$$u(t,x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\left(\int e^{itS_0(k,l;\xi,\eta)} f(k,l)\mu^l(x,l;y,t)\,dk\,dl\right)$$

where

$$S(k,l;a) = (l-k)\xi - (l^2-k^2)\eta + 4(l^3-k^3), \qquad f(k,l) = T^+(k,l) + T^-(k,l)$$

and μ^l solves the non-local Riemann Hilbert Problem

$$\mu^l = 1 + (C_+\mathcal{T}^- + C_-\mathcal{T}^+)\mu^l,$$

$$(\mathcal{T}^\pm f)(k) = \pm\int_k^{\pm\infty} e^{itS(k,l;a)} T^\pm(k,l) f(l)\,dl$$

where $C_\pm : L^2(\mathbb{R},dk) \to L^2(\mathbb{R},dk)$ are Cauchy projectors

To obtain asymptotics, we will combine stationary phase estimates with estimates on μ^l for large times

Ideas of the Proof

$$
\begin{aligned}
u(t,x,y) &= u_1(t,x,y) + u_2(t,x,y) \\
u_1(t,x,y) &= \frac{1}{\pi} \int e^{itS_0(k,l;\xi,\eta)} i(l-k) f(k,l)\, dl\, dk \\
u_2(t,x,y) &= \int e^{itS_0(k,l;\xi,\eta)} i(k-l) f(k,l) (\mu^l(x,l;y,t) - 1)\, dl\, dk \\
&\quad + \int e^{itS_0(k,l;\xi,\eta)} f(k,l) \frac{\partial \mu^l}{\partial x}(l,x;y,t)\, dl\, dk
\end{aligned}
$$

The u_1 term is a “local” term, the u_2 term is a “nonlocal” terms.

In the local term, we can “almost” use stationary phase and exploit the behavior of the phase function in the three regions.

In the nonlocal term, note that

$$
e^{itS_0(k,l;\xi,\eta)} = e^{-it(k\xi - k^2\eta + 4k^3)} e^{it(l\xi - l^2\eta + 4l^3)}
$$

We will obtain L^2 estimates for $\mu^l - 1$ and $\partial \mu^l / \partial x$ (l variable) and exploit stationary phase (k variable).

Ideas of the Proof - Local Term

We can rewrite

$$u_1(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;a)} i(l-k)(\widetilde{T}^{+}(k,l) + \widetilde{T}^{-}(k,l))\,dl\,dk$$

where

$$S(k,l;a) = 12((l-k)a + \frac{1}{3}(l^3 - k^3)).$$

Recall the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int e^{i(sx+s^3/3)}\,ds$$

and note that

$$\frac{1}{\sqrt{2\pi}}\int e^{-i\xi l} e^{12it(la+l^3/3)}\,dl = \frac{\sqrt{2\pi}}{(12t)^{\frac{1}{3}}}\,\mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a - \frac{\xi}{12t}\right)\right)$$

and the estimate

$$|\,\mathrm{Ai}(z)| \lesssim (1+|z|)^{-\frac{1}{2}}$$

holds.

Ideas of the Proof –Local Term

$$u_1(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;a)} i(l-k)(\widetilde{T}^+(k,l) + \widetilde{T}^-(k,l))\, dl\, dk$$

where

$$S(k,l;a) = 12(al - l^3/3) - 12(ak - k^3/3).$$

(i) For $a > 0$, we can use integration by parts to obtain $o(t^{-1})$ decay

(ii) For $|a| < 1$, we can use Airy asymptotics to obtain $\mathcal{O}\left(t^{-\frac{2}{3}}\right)$ decay

(iii) For $a < 0$, we can Fourier transform in one of the two variables and the cubic phase transforms to an Airy function with $\mathcal{O}\left(t^{-\frac{1}{2}}\right)$ decay. We can use similar arguments for the other integration involving partial Airy-type integrals of the form

$$\int_k^\infty e^{12it(al+l^3/3)}\, dl.$$

Ideas of the Proof - Nonlocal Term

$$u_2(t,x,y) = \int e^{itS_0(k,l;\xi,\eta)} i(k-l) f(k,l)(\mu^l(x,l;y,t)-1)\,dl\,dk$$
$$+ \int e^{itS_0(k,l;\xi,\eta)} f(k,l) \frac{\partial \mu^l}{\partial x}(l,x;y,t)\,dl\,dk$$

Recall $\mu^l = 1 + C_T \mu^l$ where

$$C_T f = C_+ \mathcal{T}^- f + C_- \mathcal{T}^+ f, \qquad \mathcal{T}^\pm f(l) = \int e^{itS_0(k,l;\xi,\eta)} T^\pm(l,l') f(l')\,dl'$$

One can show that, for small data $\|C_T\|_{L^2 \to L^2} < \frac{1}{2}$ so that

$$\mu^l - 1 = (I - C_T)^{-1} C_T(1)$$
$$\frac{\partial \mu^l}{\partial x} = (I - C_T)^{-1} \left(C_{\partial T/\partial x}\right)(\mu^l - 1) + (I - C_T)^{-1} C_{\partial T/\partial x}(1)$$

To obtain decay of the L^2_l norm in time, it suffices to estimate

$$\|C_T(1)\|_{L^2_l}, \quad \|C_{\partial T/\partial x}(1)\|_{L^2_l}$$

Ideas of the Proof - Nonlocal Term

To obtain decay of the L_l^2 norm in time, it suffices to estimate $\|C_T(1)\|_{L_l^2}$, $\|C_{\partial T/\partial x}(1)\|_{L_l^2}$ where

$$C_T f = C_+ \mathcal{T}^- f + C_- \mathcal{T}^+ f,$$

$$\mathcal{T}^\pm f(l) = \int e^{itS_0(l,l';\xi,\eta)} T^\pm(l,l') f(l')\, dl'$$

Since $C_\pm$ are isometries of L^2, it is enough to estimate $\mathcal{T}^\pm(1)$ and $\frac{\partial}{\partial x}\mathcal{T}^\pm(1)$
For example, by a change of variables

$$\mathcal{T}^+(1) = \int_l^\infty e^{itS(l,l';a)} T^+(l,l')\, dl'$$

$$\frac{\partial}{\partial x}\mathcal{T}^+(1) = \int_l^\infty e^{itS(l,l';a)} i(l-l')\mathcal{T}^+(l,l')\, dl'$$

We can use Fourier theory again provided we can estimate Fourier transforms ("almost" Airy functions) of the form

$$\int_l^\infty e^{-i\xi l'} e^{12it(al'+(l')^3/3)}\, dl'$$

Ideas of the Proof - Nonlocal Term

We obtain the following estimates:

$$\left\| \mu^l(x, \cdot; y, t) - 1 \right\|_{L_l^2} \lesssim \begin{cases} t^{-1}, & a > \delta > 0 \\ t^{-\frac{1}{3}}, & |a| \leq \delta \\ t^{-\frac{1}{2}}, & a < -\delta < 0 \end{cases}$$

$$\left\| \frac{\partial \mu^l}{\partial x}(x, \cdot; y, t) \right\|_{L_l^2} \lesssim \begin{cases} t^{-1}, & a > \delta > 0 \\ t^{-\frac{1}{3}}, & |a| \leq \delta \\ t^{-\frac{1}{2}}, & a < -\delta < 0 \end{cases}$$

Using stationary phase techniques in the integration over k, we can obtain initial decay of

$$\begin{cases} o(1), & a > \delta > 0, \\ t^{-\frac{1}{3}}, & |a| \leq \delta, \\ t^{-\frac{1}{2}}, & a < \delta < 0 \end{cases}$$

which gives the desired result in the non-scattering, transition, and scattering regions.

Remarks

1. These are, to our knowledge, the first results on pointwise asymptotics of $u(t, x, y)$ for KP I
2. The estimates we obtain on solutions of $\mu^l(l, x; t, y)$ are far sharper than previous estimats in the literature
3. We are able to get pointwise leading asymptotics for the local term, but we can't (yet) get pointwise leading asymptotics for the non-local term
4. We obtain asymptotics in the regions not covered by Manakov, Santini, and Takhtajan and with less regularity assumed on the initial data