MA 114 Worksheet #22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?
   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?
   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?
   (d) Represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{c^2} = 1$ with parametric equations.
   (e) Do the two sets of parametric equations
   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$
   and
   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$
   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.
   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.
   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?
   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.
   (a) $x = \sqrt{t}$, $y = 1 - t$.
   (b) $x = 3t - 5$, $y = 2t + 1$.
   (c) $x = \cos(t)$, $y = \sin(t)$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.
   (a) $y = x^3$ from $x = 0$ to $x = 2$.
   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

6. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.

(b) $x = \cos(\theta) + \sin(2\theta)$, $y = \cos(\theta)$, at $\theta = \pi/2$.

7. For the following parametric curve, find dy/dx.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.

8. Find d^2y/dx^2 for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \dfrac{1}{t}$, $0 < t \leq \pi$.

9. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.

10. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius r centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define θ to be the angle formed by the x-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
    (a) Draw a picture and label θ.
    (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$.
    (c) Find the length of the involute for $0 \leq \theta \leq 2\pi$.