

1 Lecture 09: The intermediate value theorem

- The intermediate value theorem
- Examples
- The bisection method

1.1 The intermediate value theorem

Example. Does the equation $x = \cos(x)$ have a solution?

This is an example of an equation that is easy to write down, but there is no simple formula that gives the solution. If we sketch a graph, we see that at 0, $\cos(0) = 1 > 0$ and at $\pi/2$, $\cos(\pi/2) = 0 < \pi/2$. Thus, we expect that the graphs cross somewhere in between. This idea is given a careful statement in the intermediate value theorem.

Theorem 1 (The intermediate value theorem) *Suppose that f is a continuous function on a closed interval $[a, b]$ with $f(a) \neq f(b)$. If M is between $f(a)$ and $f(b)$, then there is a number c in the interval (a, b) so that $f(c) = M$.*

With this we can give a careful solution to the opening example.

Solution. Let $f(x) = x - \cos(x)$. The function f is continuous everywhere. We have $f(0) = -1$ and $f(\pi/2) = \pi/2$. Since 0 lies between $f(0)$ and $f(\pi/2)$, there must be a number c with $0 < c < \pi/2$ and $f(c) = 0$ or $c = \cos(c)$. ■

Example. Is there a solution to the equation

$$\frac{3x + 1}{1 + x} = 3.$$

Why is the intermediate value theorem not useful to answer this question?

Solution. If we try to solve this equation, we have

$$\begin{aligned} 3x + 1 &= 3(x + 1) \\ 3x + 1 &= 3x + 3 \\ 1 &= 3. \end{aligned}$$

And we are stuck.

If we consider the function $f(x) = (3x + 1)/(1 + x)$, we have $f(0) = 1$ and $f(-2) = 5$, thus we might hope to find a solution in the interval $[-2, 0]$. However, the function f is not continuous on this interval—it is undefined at -1 and thus cannot be continuous. ■

The square root is a very familiar function, however we have not given any explanation about why square roots exist. The intermediate value theorem provides a way to do this.

Example. Show that there is a number so that $a^2 = 4$.

Show that there is a number $a > 0$ so that $a^2 = 2$.

Can you show that a is between 1.4 and 1.5?

Solution. The first question is easy. We may let $a = 2$.

For the second question, we observe that $f(x) = x^2$ is continuous on the interval $[1, 2]$, $f(1) = 1$ and $f(2) = 4$. Thus, there must be a value a in the interval $(1, 2)$ so that $f(a) = 2$.

To show that a is in the interval $(1.4, 1.5)$, we only need to observe that $1.4^2 = 1.96$ and $1.5^2 = 2.25$ and thus there is a solution of $f(a) = 2$ in the interval $(1.4, 1.5)$. ■

1.2 The bisection method

This theorem is useful in that it tells us that we can solve equations. Is there some way that we can improve our solution? The following procedure allows us to improve our solution. We continue to consider the equation $f(x) = x - \cos(x) = 0$. We know there is a solution in the interval $(0, \pi/2)$. If we try $c = \pi/4$, then we have $f(\pi/4) = \pi/4 - \sqrt{2}/2 \approx 0.078$. Thus we have $f(0) < 0$ and $f(\pi/4) > 0$. Continuing by testing the mid-point of the new interval, we can find a better approximation to the solution of $f(x) = 0$. This is known as the bisection method.

1. Begin with a continuous function on a closed interval $[a, b]$ with $f(a)$ and $f(b)$ of different signs. Our goal is to approximate a solution of the equation $f(x) = 0$.
2. Test f at the midpoint $m = (a + b)/2$. If $f(m) = 0$, we are very lucky and have found a solution of $f(x) = 0$. Otherwise, since $f(a)$ and $f(b)$ are of different signs, we must have that either the pair $f(a)$ and $f(m)$ are of different signs or the pair $f(m)$ and $f(b)$ are of different signs. We have found an interval of length $(b - a)/2$ which contains a solution of $f(x) = 0$.
3. Repeat.

Continuing with our example,

Example. Find an interval of length less than $1/4$ which contains a solution of $x = \cos(x)$.

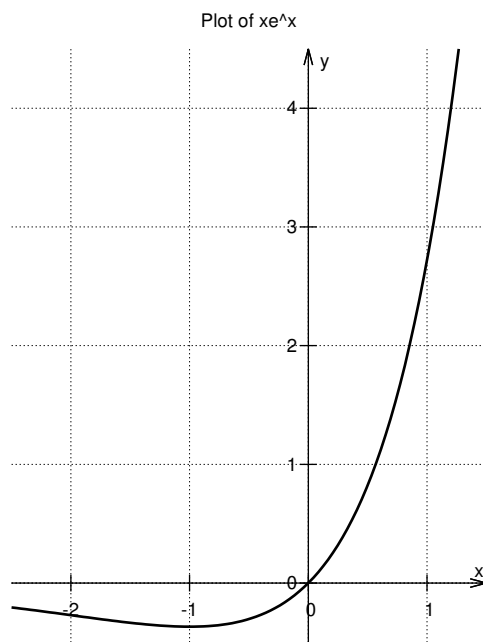
Solution. We summarize our work in a table.

$[a, b]$	Midpoint, m	$a - \cos(a)$	$m - \cos(m)$	$b - \cos(b)$
$[0, \pi/2]$	$\pi/4$	-1	0.078	$\pi/2$
$[0, \pi/4]$	$\pi/8$	-1	-0.53	0.078
$[\pi/8, \pi/4]$	$3\pi/16$	-0.53	-0.24	0.078
$[3\pi/16, \pi/4]$	$7\pi/32$	-0.24	-0.086	0.078

Since $\pi/16 < 1/4$, the interval $[3\pi/16, \pi/4]$ is a correct answer to our problem. ■

Example. Let $f(x) = xe^x$ with domain the interval $[1, \infty)$. Convince yourself that f is one-to-one and thus f^{-1} exists. Estimate $f^{-1}(2)$.

Solution. The graph of xe^x suggests that f is increasing on $[1, \infty)$ and thus one-to-one. Eventually, we will learn another way to see this.



We need to find a solution of the equation

$$xe^x = 2$$

in the interval $[-1, \infty)$. If we let $g(x) = xe^x - 2$, then we have $g(0) = -2$ and $g(1) = e \approx 2.78$. Thus we know the solution of $xe^x = 2$ lies in the interval $(0, 1)$. Bisection, we have $g(1/2) = (1/2)e^{1/2} - 2 \approx -1.18$. Thus we have a solution of $g(x) = 0$ in $(1/2, 1)$. Continuing in this way we can find better and better approximations.

Find the next two approximations. ■

1.3 Problems

1. Find an interval in which the equation $x^3 + x = 20$ has a solution.
2. Find an interval in which the equation $\sin(x) = 7 \cos(x)$ has a solution.
3. Suppose that f is continuous on the interval $[0, 4]$ and that $f(0) = f(2) = f(4) = 1$ and $f(1) = f(3) = -1$.
 - (a) Can the equation $f(x) = 0$ have exactly one solution?
 - (b) How many solutions can the equation $f(x) = 0$ have?

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