1 The fundamental theorems of calculus.

- The fundamental theorems of calculus.
- Evaluating definite integrals.
- The indefinite integral-a new name for anti-derivative.
- Differentiating integrals.

Theorem 1 *Suppose f is a continuous function on $[a,b]$.*
(FTC I) If $g(x) = \int_a^x f(t)\,dt$, then $g' = f$.
(FTC II) If F is an anti-derivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example. Compute

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt.$$

Compute

$$\int_0^3 x^3\,dx.$$

Proof. An idea of the proofs. FTC I:
Write

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

We will show

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

The reader should write out a similar argument for the limit from the below.

If f is continuous, then f has maximum and minimum values M_h and m_h on the interval $[x, x+h]$. Using the order property of the integral,

$$m_h \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq M_h.$$

As h tends to 0, we have $\lim_{h\to 0^+} M_h = \lim_{h\to 0^+} m_h = f(x)$ since f is continuous. It follows that

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x).$$

FTC II:

We know from FTCI that f has one anti-derivative, $\int_a^x f(t)\,dt$. We let

$$G(x) = \int_a^x f(t)\,dt - F(x)$$

where F is some anti-derivative as in FTC II. The derivative of G, $G'(x) = f(x) - f(x) = 0$ for all x in (a, b). This uses FTC I and the hypothesis that F is an anti-derivative of f. Since the derivative of G is identically zero, we can conclude that G is a constant.

If we set $x = a$ in the definition of G, we find $G(a) = -F(a)$ so that we can conclude the constant is $-F(a)$. If we set $x = b$ in the definition of G, then we conclud

$$-F(a) = \int_a^b f(t)\,dt - F(b).$$

Adding $F(b)$ to both sides give the conclusion of FTC II. ∎

1.1 Indefinite integrals.

We use the symbol

$$\int f(x)\,dx$$

to denote the indefinite integral or anti-derivative of f.

The indefinite integral is a function. The definite integral is a number. According FTC II, we can find the (numerical) value of a definite integral by evaluating the indefinite integral at the endpoints of the integral. Since this procedure happens so often, we have a special notation for this evaluation.

$$F(x)|_{x=a}^{b} = F(b) - F(a).$$

Example. Find

$$xa|_{x=a}^{b} \qquad \text{and} \qquad xa|_{a=x}^{y}$$

Solution.

$$ba - a^2 \qquad xy - x^2$$

∎

According to FTC I, anti-derivatives exist provided f is continuous.

The box on page 351 should be memorized. (In fact, you should already have memorized this information when we studied derivatives in Chapter 3 and when we studied anti-derivatives in Chapter 4.)

Example. Verify

$$\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2).$$

Solution. According to the definition of anti-derivative, we need to see if

$$\frac{d}{dx}\frac{1}{2}\sin(x^2) = x\cos(x^2).$$

This holds, by the chain rule. ∎

1.2 Computing integrals.

The main use of FTC II is to simplify the evaluation of integrals.

We give a few examples.

Example. a) Compute

$$\int_0^\pi \sin(x)\,dx.$$

b) Compute

$$\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx.$$

Solution. a) Since $\frac{d}{dx}(-\cos(x)) = \sin(x)$, we have $-\cos(x)$ is an anti-derivative of $\sin(x)$. Using the second part of the fundamental theorem of calculus gives,

$$\int_0^\pi \sin(x)\,dx = -\cos(x)|_{x=0}^{\pi} = 2.$$

b) We first find an anti-derivative. As the indefinite integral is linear, we write

$$\int \frac{2x^2+1}{\sqrt{x}}\,dx = \int 2x^{3/2} + x^{-1/2}\,dx \quad = \quad 2\int x^{3/2}\,dx + \int x^{-1/2}\,dx = \frac{4}{5}x^{5/2} + 2x^{1/2} + C.$$

With this anti-derivative, we may then use FTC II to find

$$\begin{aligned}
\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx &= \left.\frac{4}{5}x^{5/2} + 2x^{1/2}\right|_{x=1}^{4} \\
&= \frac{4}{5}4^{5/2} + 24^{1/2} - (\frac{4}{5} + 2) \\
&= 128/5 + 20/5 - (4/5 + 10/5) \\
&= 134/5.
\end{aligned}$$

∎

Here, is a more involved example that illustrates the progress we have made.

Example. Find

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(k/n).$$

Solution. We recognize that

$$\frac{1}{n}\sum_{k=1}^{n}\sin(k/n)$$

is a Riemann sum for an integral. The points x_k, $k = 0, \dots, n$ divide the interval $[0, 1]$ into n equal sub-intervals of length $1/n$. Thus, we may write the limit as an integral

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sin(k/n) = \int_0^1 \sin(x)\,dx.$$

To evaluate the resulting integral, we use FTCII. An anti-derivative of $\sin(x)$ is $-\cos(x)$, thus

$$\int_0^1 \sin(x)\,dx = -\cos(x)|_{x=0}^{1} = 1 - \cos(1).$$

∎

1.3 Differentiating integrals.

FTC I plays an important role in the proof of FTC II. It is also used to find the derivatives of integrals.

Example. Find

$$\frac{d}{dx}\int_0^x \sin(t^2)\,dt \qquad \frac{d}{dx}\int_{x^2}^x \sin(t^2)\,dt \qquad \frac{d}{dx}\int_1^x \frac{1}{t}\,dt$$

Is the function $L(x) = \int_1^x \frac{1}{t}\,dt$ increasing or decreasing? Is the graph of L concave up or concave down?

1.4 The net change theorem

Since F is always an anti-derivative of F', one consequence of part II of the fundamental theorem of calculus is that if F' is continuous on the interval $[a, b]$, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

This helps us to understand some common physical interpretations of the integral.

For example, if $p(t)$ denotes the position of an object. More precisely, if an object is moving along a line and p gives the number of meters the object lies to the right of a reference point, then $p' = v$ is the velocity of the object. The definite integral

$$p(b) - p(a) = \int_a^b v(t)\,dt \qquad (1)$$

denotes the net change in position of the object during the interval $[a, b]$. Note that if v is measured in meters/second, then units on $v(t)dt$ would be meters/second $\times$ seconds so the equation (1) is a sophisticated version of the familiar fact that distance $=$ rate $\times$ time.

To give a less familiar example, suppose we have a rope whose thickness varies along its length. Fix one end of the rope to measure from and let $m(x)$ denote the mass in kilograms of the rope from 0 to x meters along the rope. If we take the derivative, $\frac{dm}{dx} = \lim_{h\to 0} m(x+h) - m(x)h$, then this represents an average mass of the rope near x whose units are kilograms/meter. If we integrate this linear density and observe that $m(0) = 0$, then we recover the mass

$$m(x) = \int_0^x \frac{dm}{dx}\, dx.$$

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