Lecture notes: harmonic analysis

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Contents

Preface

These notes are intended for a course in harmonic analysis on $\mathbf{R}^n$ which was offered to graduate students at the University of Kentucky in Spring of 2001. The background for this course is a course in real analysis which covers measure theory and the basic facts of life related to L^p spaces. The students who were subjected to this course had studied from *Measure and integral* by Wheeden and Zygmund and *Real analysis: a modern introduction*, by Folland.

Much of the material in these notes is taken from the books of Stein *Singular integrals and differentiability properties of functions,* [29] and *Harmonic analysis* [30] and the book of Stein and Weiss, *Fourier analysis on Euclidean spaces* [31]. The monograph of Loukas Grafakos, *Classical and modern Fourier analysis* [14] provides an excellent treatment of the Fourier analysis in the first half of these notes.

The exercises serve a number of purposes. They illustrate extensions of the main ideas. They provide a chance to state simple results that will be needed later. They occasionally give interesting problems.

These notes are at an early stage and far from perfect. Please let me know of any errors.

Participants in the 2008 version of the course include Jun Geng, Jay Hineman, Joel Kilty, Julie Miker, Zhongyi Nie, Michael Shaw, and Justin Taylor. Their contributions to improving these notes are greatly appreciated.

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The latex source is available from the author.

Chapter 1

The Fourier transform on L^1

In this chapter, we define the Fourier transform and give the basic properties of the Fourier transform of an $L^1(\mathbf{R}^n)$ function. Recall that $L^1(\mathbf{R}^n)$ is the space of Lebesgue measurable functions for which the norm $\|f\|_1 = \int_{\mathbf{R}^n} |f(x)|\,dx$ is finite. For $0 < p < \infty$, $L^p(\mathbf{R}^n)$ denotes the space of Lebesgue measurable functions for which the norm $\|f\|_p = (\int_{\mathbf{R}^n} |f(x)|^p\,dx)^{1/p}$ is finite. When $p = \infty$, the space $L^\infty(\mathbf{R}^n)$ is the collection of measurable functions which are essentially bounded. For $1 \leq p \leq \infty$, the space $L^p(\mathbf{R}^n)$ is a Banach space. We recall that a vector space V over $\mathbf{C}$ with a function $\|\cdot\|$ is called a normed vector space if $\|\cdot\| : V \to [0, \infty)$ and satisfies

$$\begin{aligned}
\|f+g\| &\leq \|f\| + \|g\|, \qquad f, g \in V \\
\|\lambda f\| &= |\lambda|\|f\|, \qquad f \in v,\ \lambda \in \mathbf{C} \\
\|f\| &= 0, \qquad \text{if and only if } f = 0.
\end{aligned}$$

A function $\|\cdot\|$ which satisfies these properties is called a *norm*. If $\|\cdot\|$ is a norm, then $\|f - g\|$ defines a metric. A normed vector space $(V, \|\cdot\|)$ is called a *Banach space* if V is complete in the metric defined using the norm. Throughout these notes, functions are assumed to be complex valued.

1.1 Definition and symmetry properties

We define the Fourier transform. In this definition, $x \cdot \xi$ is the inner product of two elements of $\mathbf{R}^n$, $x \cdot \xi = \sum_{j=1}^n x_j \xi_j$.

Definition 1.1 *If $f \in L^1(\mathbf{R}^n)$, then the* Fourier transform *of f, $\hat{f}$, is a function defined* *on* $\mathbf{R}^n$ *and is given by*

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}\, dx.$$

The Fourier transform is a continuous map from L^1 to the bounded continuous functions on $\mathbf{R}^n$.

Proposition 1.2 *If* $f \in L^1(\mathbf{R}^n)$*, then* $\hat{f}$ *is continuous and*

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

Proof. The estimate follows since $e^{-ix\cdot\xi}$ is of modulus 1. Let $\{\xi^j\}$ be a sequence in $\mathbf{R}^n$ with $\lim_{j\to\infty}\xi^j = \xi$, then we have $\lim_{j\to\infty} e^{-ix\cdot\xi^j} f(x) = f(x)e^{-ix\cdot\xi}$ and $|e^{-ix\cdot\xi^j} f(x)| \leq |f(x)|$. By the Lebesgue dominated convergence theorem, we have $\lim_{j\to\infty}\hat{f}(\xi^j) \to \hat{f}(\xi)$. ∎

The inequality in the conclusion of Proposition 1.2 is equivalent to the continuity of the map $f \to \hat{f}$. This is an application of the conclusion of the following exercise.

Exercise 1.3 *A linear map* $T : V \to W$ *between normed vector spaces is continuous if and only if there exists a constant* C *so that*

$$\|Tf\|_W \leq C\|f\|_V.$$

In the following proposition, we use $A^{-t} = (A^{-1})^t$ for the transpose of the inverse of an $n\times n$ matrix, A.

Exercise 1.4 *Show that if* A *is an* $n \times n$ *invertible matrix, then* $(A^{-1})^t = (A^t)^{-1}$.

Exercise 1.5 *Show that* A *is an* $n \times n$ *matrix, then* $Ax \cdot y = x \cdot A^t y$.

Proposition 1.6 *If* A *is an* $n \times n$ *invertible matrix with real entries, then*

$$\widehat{f\circ A} = |\det A|^{-1}\hat{f}\circ A^{-t}.$$

Proof. If we make the change of variables, $y = Ax$ in the integral defining $\widehat{f\circ A}$, then we obtain

$$\begin{aligned}
\widehat{f\circ A}(\xi) &= \int_{\mathbf{R}^n} f(Ax)e^{-ix\cdot\xi}\, dx \\
&= |\det A|^{-1}\int_{\mathbf{R}^n} f(y)e^{-iA^{-1}y\cdot\xi}\, dy \\
&= |\det A|^{-1}\int_{\mathbf{R}^n} f(y)e^{-iy\cdot A^{-t}\xi}\, dy.
\end{aligned}$$

∎

If we set $f_\epsilon(x) = \epsilon^{-n} f(x/\epsilon)$ for $\epsilon > 0$, then a simple application of Proposition 1.6 gives

$$\hat{f}_\epsilon(\xi) = \hat{f}(\epsilon\xi). \tag{1.7}$$

Recall that an orthogonal matrix is an $n \times n$-matrix with real entries which satisfies $O^t O = I_n$ where I_n is the $n \times n$ identity matrix. Such matrices are clearly invertible since $O^{-1} = O^t$. The group of all such matrices is usually denoted by $O(n)$.

Corollary 1.8 *If $f \in L^1(\mathbf{R}^n)$ and O is an orthogonal matrix, then $\hat{f} \circ O = \widehat{f \circ O}$.*

Exercise 1.9 *If $x \in \mathbf{R}^n$, show that there is an orthogonal matrix O so that $Ox = (|x|, 0, \dots, 0)$.*

Exercise 1.10 *Let A be an $n \times n$ matrix with real entries. Show that A is orthogonal if and only if $Ax \cdot Ax = x \cdot x$ for all $x \in \mathbf{R}^n$.*

We say that function f defined on $\mathbf{R}^n$ is *radial* if there is a function F on $[0, \infty)$ so that $f(x) = F(|x|)$. Equivalently, a function is radial if and only if $f(Ox) = f(x)$ for all orthogonal matrices O.

Corollary 1.11 *Suppose that f is in L^1 and f is radial, then $\hat{f}$ is radial.*

Proof. We fix ξ in $\mathbf{R}^n$ and choose O so that $O\xi = (|\xi|, 0, \dots, 0)$. Since $f \circ O = f$, we have that $\hat{f}(\xi) = \widehat{f \circ O}(\xi) = \hat{f}(O\xi) = \hat{f}(|\xi|, 0, \dots, 0)$. ∎

We shall see that many operations that commute with translations can be expressed as multiplication operators using the Fourier transform. One important operation which commutes with translations is differentiation. Below we shall see how to display this operation as a multiplication operator. As our first example of this principle, we will see that the operation of translation by h (which surely commutes with translations) corresponds to multiplying the Fourier transform by $e^{ih\cdot\xi}$. We will use τ_h to denote translation by $h \in \mathbf{R}^n$, $\tau_h f(x) = f(x + h)$.

Exercise 1.12 *If f is a differentiable function on $\mathbf{R}^n$, show that*

$$\frac{\partial}{\partial x_j} \tau_h f = \tau_h \frac{\partial}{\partial x_j} f.$$

Proposition 1.13 *If f is in $L^1(\mathbf{R}^n)$, then*

$$\widehat{\tau_h f}(\xi) = e^{ih\cdot\xi}\hat{f}(\xi).$$

Also,

$$(e^{ix\cdot h}f)\hat{} = \tau_{-h}(\hat{f}). \tag{1.14}$$

Proof. We change variables $y = x + h$ in the integral

$$\widehat{\tau_h f}(\xi) = \int f(x+h)e^{-ix\cdot\xi}\,dx = \int f(y)e^{-i(y-h)\cdot\xi}\,dy = e^{ih\cdot\xi}\hat{f}(\xi).$$

The proof of the second identity is just as easy and is left as an exercise. ∎

Example 1.15 *If $I = \{x : |x_j| < 1\}$, then the Fourier transform of $f = \chi_I$ is easily computed,*

$$\hat{f}(\xi) = \prod_{j=1}^{n}\int_{-1}^{1} e^{ix_j\xi_j}\,dx_j = \prod_{j=1}^{n}\frac{2\sin\xi_j}{\xi_j}.$$

In the next exercise, we will need to write integrals in polar coordinates. For our purposes, this means that we have a Borel measure σ on the *sphere*, $\mathbf{S}^{n-1} = \{x' \in \mathbf{R}^n : |x'| = 1\}$ so that

$$\int_{\mathbf{R}^n} f(x)\,dx = \int_0^\infty \int_{\mathbf{S}^{n-1}} f(rx')\,d\sigma(x')r^{n-1}\,dr.$$

Exercise 1.16 *If $B_r(x) = \{y : |x-y| < r\}$ and $f = \chi_{B_1(0)}$, compute the Fourier transform $\hat{f}$.*

Hints: 1. Since f is radial, it suffices to compute $\hat{f}$ at $(0,\dots,r)$ for $r > 0$. 2. Write the integral over the ball as an iterated integral where we integrate with respect to $x' = (x_1,\dots,x_{n-1})$ and then with respect to x_n. 3. You will need to know the volume of a ball, see exercise 1.30 below. 4. At the moment, we should only complete the computation in 3 dimensions (or odd dimensions, if you are ambitious). In even dimensions, the answer cannot be expressed in terms of elementary functions. See Chapter 14 for the answer in even dimensions. One possible answer is

$$\hat{f}(\xi) = \frac{\omega_{n-2}}{n-1}\int_{-1}^{1} e^{-it|\xi|}(1-t^2)^{(n-1)/2}\,dt.$$

Here, ω_{n-2} is the surface area of the unit ball as defined in Exercise 1.30.

Theorem 1.17 *(Riemann-Lebesgue) If f is in $L^1(\mathbf{R}^n)$, then*

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

Proof. We let $X \subset L^1(\mathbf{R}^n)$ be the collection of functions f for which $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$. It is easy to see that X is a vector space. Thanks to Proposition 1.2, X is closed in the L^1-norm. According to Example 1.15, Proposition 1.13 and Proposition 1.6 the characteristic function of every rectangle is in X. Since finite linear combinations of characteristic functions of rectangles are dense in L^1, $X = L^1(\mathbf{R}^n)$. ∎

Combining the Riemann-Lebesgue Lemma and Proposition 1.2, we can show that the image of $L^1(\mathbf{R}^n)$ under the Fourier transform is contained in $C_0(\mathbf{R}^n)$, the continuous functions on $\mathbf{R}^n$ which vanish at infinity. This containment is strict. We will see that the Fourier transform of the surface measure on the sphere $\mathbf{S}^{n-1}$ is in $C_0(\mathbf{R}^n)$. It is a difficult and unsolved problem to describe the image of L^1 under the Fourier transform.

One of our goals is to relate the properties of f to those of $\hat{f}$. There are two general principles which we will illustrate below. These principles are: *If f is smooth, then $\hat{f}$ decays at infinity* and *If f decays at infinity, then $\hat{f}$ is smooth.* We have already seen several examples of these principles. Proposition 1.2 asserts that if f is in L^1, which requires decay at infinity, then $\hat{f}$ is continuous. The Riemann-Lebesgue Lemma tells us that if f is in L^1, and thus is smoother than the distributions to be discussed below, then $\hat{f}$ has limit 0 at infinity. The propositions below give further illustrations of these principles.

Proposition 1.18 *If f and $x_j f$ are in L^1, then $\hat{f}$ is differentiable and the derivative is given by*

$$i\frac{\partial}{\partial \xi_j}\hat{f} = \widehat{x_j f}.$$

Furthermore, we have

$$\|\frac{\partial f}{\partial \xi_j}\|_\infty \leq \|x_j f\|_1.$$

Proof. Let $h \in \mathbf{R}$ and suppose that e_j is the unit vector parallel to the x_j-axis. Using the mean-value theorem from calculus, one obtains that

$$\left|\frac{e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}}{h}\right| \leq |x_j|.$$

Our hypothesis that $x_j f$ is in L^1 allows us to use the dominated convergence theorem to bring the limit inside the integral and compute the partial derivative

$$\frac{\partial \hat{f}(\xi)}{\partial \xi_j} = \lim_{h\to 0} \int \frac{e^{-ix\cdot(\xi+he_j)} - e^{-ix\cdot\xi}}{h} f(x)\,dx = \int (-ix_j) e^{-ix\cdot\xi} f(x)\,dx.$$

The estimate follows immediately from the formula for the derivative. ∎

Note that the notation in the previous proposition is not ideal since the variable x_j appears multiplying f, but not as the argument for f. One can resolve this problem by decreeing that the symbol x_j stands for the multiplication operator $f \to x_j f$ and the jth component of x.

For the next proposition, we need an additional definition. We say f has a partial derivative with respect to x_j in the L^p sense if f is in L^p and there exists a function $\partial f/\partial x_j$ in $L^p(\mathbf{R}^n)$ so that

$$\lim_{h\to 0} \|\frac{1}{h}(\tau_{he_j} f - f) - \frac{\partial f}{\partial x_j}\|_p = 0.$$

Proposition 1.19 *If f is differentiable with respect to x_j in the L^1-sense, then*

$$i\xi_j \hat{f} = \widehat{\frac{\partial f}{\partial x_j}}.$$

Furthermore, we have

$$\|\xi_j \hat{f}\|_\infty \leq \left\| \frac{\partial f}{\partial x_j} \right\|_1 .$$

Proof. Let $h > 0$ and let e_j be a unit vector in the direction of the x_j - axis. Since the difference quotient converges in L^1, we have

$$\int_{\mathbf{R}^n} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_j}(x)\,dx = \lim_{h\to 0} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} \frac{f(x+he_j) - f(x)}{h}\,dx.$$

In the last integral, we can make a change of variables $y = x + he_j$ to move the difference operator to the exponential function

$$\int_{\mathbf{R}^n} \frac{e^{-i(x-he_j)\cdot\xi} - e^{-ix\cdot\xi}}{h} f(x)\,dx.$$

Since the difference quotient of the exponential converges pointwise and boundedly in x to $i\xi_j e^{-ix\cdot\xi}$, we can use the dominated convergence theorem to obtain $\widehat{\partial f}/\partial x_j = i\xi_j \hat{f}$. ∎

Finally, our last result on translation invariant operators involves convolution. Recall that if f and g are measurable functions on $\mathbf{R}^n$, then the convolution is defined by

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy$$

provided the integral on the right is defined for a.e. x.

Some of the basic properties of convolutions are given in the following exercises. The solutions can be found in most real analysis texts.

Exercise 1.20 *If f is in L^1 and g is in L^p, with $1 \le p \le \infty$, show that $f * g(x)$ is defined a.e. and*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Exercise 1.21 *Show that the convolution is commutative. If $f * g(x)$ is given by a convergent integral, then*

$$f * g(x) = g * f(x).$$

If f, g and h are in L^1, show that convolution is associative

$$f * (g * h) = (f * g) * h.$$

Hint: Change variables.

Exercise 1.22 *The map $f \to f * g$ commutes with translations:*

$$\tau_h(f * g) = (\tau_h f) * g.$$

Exercise 1.23 *(Young's convolution inequality) If the exponents p, q and s satisfy $1/s = 1/p + 1/q - 1$, then*

$$\|f * g\|_s \le \|f\|_p \|g\|_q.$$

The following proposition shows that the image of L^1 under the Fourier transform is an algebra under pointwise multiplication. This algebra is usually called the *Wiener algebra.*

Proposition 1.24 *If f and g are in L^1, then*

$$(f * g)\hat{} = \hat{f}\hat{g}.$$

Proof. The proof is an easy application of Fubini's theorem. ∎

Next, we calculate a very important Fourier transform. The function W in the next proposition gives (a multiple of) the Gaussian probability distribution.

Proposition 1.25 *Let $W(x)$ be defined by $W(x) = \exp(-|x|^2/4)$. Then*

$$\hat{W}(\xi) = (\sqrt{4\pi})^n \exp(-|\xi|^2).$$

Proof. We use Fubini's theorem to write $\hat{W}$ as a product of one-dimensional integrals

$$\int_{\mathbf{R}^n} e^{-|x|^2/4} e^{-ix\cdot\xi}\, dx = \prod_{j=1}^{n} \int_{\mathbf{R}} e^{-x_j^2/4} e^{-ix_j\xi_j}\, dx_j.$$

To evaluate the one-dimensional integral, we use complex analysis. We complete the square in the exponent for the first equality and then use Cauchy's integral theorem to shift the contour of integration in the complex plane. This gives

$$\int_{\mathbf{R}} e^{-x^2/4} e^{-ix\xi}\, dx = e^{-|\xi|^2} \int_{\mathbf{R}} e^{-(\frac{x}{2}+i\xi)^2}\, dx = e^{-|\xi|^2} \int_{\mathbf{R}} e^{-|x|^2/4}\, dx = \sqrt{4\pi} e^{-|\xi|^2}.$$

∎

Exercise 1.26 *Carefully justify the shift of contour in the previous proof.*

Exercise 1.27 *Establish the formula*

$$\int_{\mathbf{R}^n} e^{-\pi|x|^2}\, dx = 1$$

which was used above. a) First consider $n = 2$ and write the integral over $\mathbf{R}^2$ in polar coordinates.

b) Deduce the general case from this special case.

Exercise 1.28 *In this exercise, we give an alternate proof of Proposition 1.25 in the case $n = 1$.*

Let $\phi(\xi) = \int_{\mathbf{R}} e^{-x^2/4} e^{-ix\cdot\xi}\, dx$. Differentiate under the integral sign and use Proposition 1.19 to show that $\phi'(\xi) = -2\xi\phi(\xi)$. Use exercise 1.27 to compute $\phi(0)$. Thus ϕ is a solution of the initial-value problem

$$\begin{cases} \phi'(\xi) = -2\xi\phi(\xi) \\ \phi(0) = (4\pi)^{1/2} \end{cases}$$

One solution of this initial-value problem is given by $(4\pi)^{1/2}e^{-\xi^2}$.

To establish uniqueness, suppose that ψ is a solution of $\psi'(\xi) = -2\xi\psi(\xi)$ and differentiate to show that the $\psi(\xi)e^{\xi^2}$ is constant.

In the next exercise, we use the Γ function, defined for $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

Exercise 1.29 *a) Use exercise 1.27 to find* $\Gamma(1/2)$.

b) Integrate by parts to show that $\Gamma(s+1) = s\Gamma(s)$. *Conlude that* $\Gamma(n+1) = n!$ *for* $n = 1, 2, 3, \ldots$.

c) Use the formula $\Gamma(s+1) = s\Gamma(s)$ *to extend* Γ *to the range* $\operatorname{Re} s > -1$. *Find* $\Gamma(-1/2)$.

Exercise 1.30 *a) Use the result of exercise 1.27 and polar coordinates to compute* ω_{n-1}*, the* $n-1$*-dimensional measure of the unit sphere in* $\mathbf{R}^n$*, and show that*

$$\omega_{n-1} = \sigma(\mathbf{S}^{n-1}) = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}.$$

b) Let $m(E)$ *denote the Lebesgue measure of a set in* $\mathbf{R}^n$*. Use the result of part a) and polar coordinates to find the volume of the unit ball in* $\mathbf{R}^n$*. Show that*

$$m(B_1(0)) = \omega_{n-1}/n.$$

1.2 The Fourier inversion theorem

In this section, we show how to recover an L^1-function from the Fourier transform. A consequence of this result is that we are able to conclude that the Fourier transform is injective. The proof we give depends on the Lebesgue differentiation theorem. We will discuss the Lebesgue differentiation theorem in the chapter on maximal functions, Chapter 4.

We begin with a simple lemma.

Lemma 1.31 *If* f *and* g *are in* $L^1(\mathbf{R}^n)$*, then*

$$\int_{\mathbf{R}^n} \hat{f}(x) g(x)\, dx = \int_{\mathbf{R}^n} f(x)\hat{g}(x)\, dx.$$

Proof. We consider the integral of $f(x)g(y)e^{-iy\cdot x}$ on $\mathbf{R}^{2n}$. We use Fubini's theorem to write this as an iterated integral. If we compute the integral with respect to x first, we obtain the integral on the left-hand side of the conclusion of this lemma. If we compute the integral with respect to y first, we obtain the right-hand side. ∎

We are now ready to show how to recover a function in L^1 from its Fourier transform.

Theorem 1.32 *(Fourier inversion theorem) If f is in $L^1(\mathbf{R}^n)$ and we define f_t for $t > 0$ by*

$$f_t(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-t|\xi|^2} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi,$$

then

$$\lim_{t\to 0^+} \|f_t - f\|_1 = 0$$

and

$$\lim_{t\to 0^+} f_t(x) = f(x), \qquad a.e.\ x.$$

Proof. We consider the function $g(x) = e^{-t|x|^2+iy\cdot x}$. By Proposition 1.25, (1.7) and (1.14), we have that

$$\hat{g}(x) = (2\pi)^n (4\pi t)^{-n/2} \exp(-|y-x|^2/4t).$$

Thus applying Lemma 1.31 above, we obtain that

$$\frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} e^{-t|\xi|^2}\, d\xi = \int_{\mathbf{R}^n} f(x)(4\pi t)^{-n/2} \exp(-\frac{|y-x|^2}{4t})\, dx.$$

Thus, $f_t(x)$ is the convolution of f with the Gaussian and it is known that $f_t \to f$ in L^1. That f_t converges to f pointwise a.e. is a standard consequence of the Lebesgue differentiation theorem. A proof will be given in Chapter 5. ∎

It is convenient to have a notation for the inverse operation to the Fourier transform. The most common notation is $\check{f}$. Many properties of the inverse Fourier transform follow easily from the properties of the Fourier transform and the inversion. The following simple formulae illustrate the close connection:

$$\check{f}(x) = \frac{1}{(2\pi)^n} \hat{f}(-x) \qquad (1.33)$$

$$\check{\bar{f}}(x) = \frac{1}{(2\pi)^n} \bar{\hat{f}}(x). \qquad (1.34)$$

If $\hat{f}$ is in L^1, then the limit in t in the Fourier inversion theorem can be brought inside the integral (by the dominated convergence theorem) and we have

$$\check{f}(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} f(\xi) e^{ix\cdot\xi}\, d\xi.$$

Exercise 1.35 *Prove the formulae (1.33) and (1.34) above.*

Chapter 2

Tempered distributions

In this chapter, we introduce the Schwartz space. This is a space of well-behaved functions on which the Fourier transform is invertible. One of the main interests of this space is that other interesting operations such as differentiation are also continuous on this space. Then, we are able to extend differentiation and the Fourier transform to act on the dual space. This dual space is called the space of tempered distributions. The word tempered means that in a certain sense, the distributions do not grow too rapidly at infinity. Distributions have a certain local regularity–on a compact set a distribution can be obtained by differentiating a continuous function finitely many times. Given the connection between the local regularity of a function and the growth of its Fourier transform, it seems likely that any space on which the Fourier transform acts should have some restriction on the growth at infinity.

2.1 Test functions

The main notational complication of this chapter is the use of multi-indices. A multi-index is an n-tuple of non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_n)$. For a multi-index α, we let

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

We also use this notation for partial derivatives,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

Several other related notations are

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \alpha! = \alpha_1! \ldots \alpha_n!.$$

Note that the notation of the *length* of a multi-index α, $|\alpha|$, conflicts with the standard notation for the Euclidean norm. This inconsistency is firmly embedded in analysis and I will not try to change it.

Below are a few exercises which illustrate the use of this notation.

Exercise 2.1 *The multi-nomial theorem.*

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} x^\alpha.$$

Exercise 2.2 *Show that*

$$(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma.$$

Exercise 2.3 *The Leibniz rule. If f and g have continuous derivatives of order up to k on $\mathbf{R}^n$ and α is a multi-index of length k, then*

$$\frac{\partial^\alpha (fg)}{\partial x^\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \frac{\partial^\beta f}{\partial x^\beta} \frac{\partial^\gamma g}{\partial x^\gamma}. \tag{2.4}$$

Exercise 2.5 *Show for each multi-index α,*

$$\frac{\partial^\alpha}{\partial x^\alpha} x^\alpha = \alpha!.$$

More generally, show that

$$\frac{\partial^\beta}{\partial x^\beta} x^\alpha = \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta}.$$

The right-hand side in this last equation is defined to be zero if any component of $\alpha - \beta$ is negative.

To define the Schwartz space, we define a family of semi-norms on the collection of infinitely differentiable functions on $\mathbf{R}^n$, $C^\infty(\mathbf{R}^n)$. For each pair of multi-indices α and β, we let

$$\rho_{\alpha\beta}(f) = \sup_{x\in\mathbf{R}^n} |x^\alpha \frac{\partial^\beta f}{\partial x^\beta}(x)|.$$

We say that a function f is in the *Schwartz space* on $\mathbf{R}^n$ if $\rho_{\alpha\beta}(f) < \infty$ for all α and β. This space is denoted by $\mathcal{S}(\mathbf{R}^n)$. Recall that a norm was defined in Chapter 1. If a function $\rho : V \to [0,\infty)$ satisfies $\rho(f+g) \leq \rho(f) + \rho(g)$ for all f and g in V and $\rho(\lambda f) = |\lambda|\rho(f)$, then ρ is called a *semi-norm* on the vector space V.

The Schwartz space is given a topology using the semi-norms $\rho_{\alpha\beta}$ in the following way. Let $\{\rho_j\}_{j=1}^{\infty}$ be an arbitrary ordering of the norms $\rho_{\alpha\beta}$. Let $\bar\rho_j = \min(\rho_j, 1)$ and then define

$$\rho(f-g) = \sum_{j=1}^{\infty} 2^{-j}\bar\rho_j(f-g).$$

Lemma 2.6 *The function ρ is a metric on $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$ is complete in this metric. The vector operations $(f,g) \to f+g$ and $(\lambda, f) \to \lambda f$ are continuous on $\mathcal{S}(\mathbf{R}^n)$.*

Exercise 2.7 *Prove Lemma 2.6.*

Note that our definition of the metric involves an arbitrary ordering of the norms $\rho_{\alpha\beta}$. One consequence of the next proposition, Proposition 2.8 is that the topology on $\mathcal{S}(\mathbf{R}^n)$ does not depend on the ordering of the semi-norms.

Proposition 2.8 *A set $\mathcal{O}$ is open in $\mathcal{S}(\mathbf{R}^n)$ if and only for each $f \in \mathcal{O}$, there exist finitely many semi-norms $\rho_{\alpha_i\beta_i}$ and $\epsilon > 0$, $i = 1, \ldots, N$ so that*

$$\{g : \rho_{\alpha_i\beta_i}(f-g) < \epsilon,\ i = 1, \ldots, N\} \subset \mathcal{O}.$$

We will not use this proposition, thus the proof is left as an exercise.

Exercise 2.9 *Prove Proposition 2.8 Hint: Read the proof of Proposition 2.11.*

Exercise 2.10 *The Schwartz space is an example of a Fréchet space. A* Fréchet space *is a vector space X with a countable family of semi-norms $\{\rho_j\}$. We define $\rho(f-g)$ by $\rho(f-g) = \sum 2^{-j}\bar\rho_j(f-g)$. The space X is Fréchet if ρ is a metric and if X is complete in the metric ρ.*

Show that $\mathcal{S}(\mathbf{R}^n)$ is a Fréchet space.

Proposition 2.11 *a) A linear map T from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ is continuous if and only if for each semi-norm $\rho_{\alpha\beta}$, there exists a finite collection of semi-norms $\{\rho_{\alpha_i\beta_i} : i = 1\ldots, N\}$ and a constant C so that*

$$\rho_{\alpha\beta}(Tf) \leq C\sum_{i=1}^{N}\rho_{\alpha_i\beta_i}(f).$$

b) A map u from $\mathcal{S}(\mathbf{R}^n)$ to a normed vector space V is continuous if and only if there exists a finite collection of semi-norms $\{\rho_{\alpha_i\beta_i} : i = 1\dots, N\}$ and a constant C so that

$$\|u(f)\|_V \le C \sum_{i=1}^{n} \rho_{\alpha_i\beta_i}(f).$$

Proof. To prove part a), we first suppose that $T : \mathcal{S} \to \mathcal{S}$ is continuous. Let the given semi-norm $\rho_{\alpha\beta} = \rho_N$ under the ordering used to define the metric. Then T is continuous at 0 and hence given $\epsilon = 2^{-N-1}$, there exists $\delta > 0$ so that if $\rho(g) < \delta$, then $\rho(Tg) < 2^{-N-1}$. We may choose M so that $\sum_{j=M+1}^{\infty} 2^{-j} < \delta/2$. Given f, we set

$$g = \frac{\delta}{2} \frac{f}{\sum_{j=1}^{M} 2^{-j}\rho_j(f)}.$$

The function g satisfies $\rho(g) < \delta$ and thus $\rho(Tg) < 2^{-N-1}$. This implies that $\rho_N(Tg) < 1/2$. Thus, by the homogeneity of ρ_N and T, we obtain

$$\rho_N(Tf) \le \frac{1}{\delta} \sum_{j=1}^{M} 2^{-j}\rho_j(f).$$

Now suppose that the second condition of our theorem holds and we verify that the $\epsilon - \delta$ formulation of continuity holds. Since the map T is linear, it suffices to prove that T is continuous at 0. Let $\epsilon > 0$ and then choose N so that $2^{-N} < \epsilon/2$. For each $j = 1, \dots, N$, there exists $C_j > 0$ and M_j so that

$$\rho_j(Tf) \le C_j \sum_{k=1}^{M_j} \rho_k(f).$$

If we set $M_0 = \max(M_1, \dots, M_N)$, and $C_0 = \max(C_1, \dots, C_N)$, then we have

$$\begin{aligned} \rho(Tf) &\le \sum_{j=1}^{N} 2^{-j}\rho_j(Tf) + \frac{\epsilon}{2} \\ &\le C_0 \sum_{j=1}^{N} \left(2^{-j} \sum_{k=1}^{N_0} \rho_k(f) \right) + \frac{\epsilon}{2}. \end{aligned} \tag{2.12}$$

Now we define δ by $\delta = 2^{-M_0}\min(1, \epsilon/(2M_0C_0))$. If we have $\rho(f) < \delta$, then we have $\bar{\rho}_k(f) < 1$ and $\rho_k(f) < \epsilon/(2M_0C_0)$ for $k = 1, \dots, M_0$. Hence, we have $\rho_k(f) < \epsilon/(2M_0C_0)$ for $k = 1, \dots, M_0$. Substituting this into the inequality (2.12) above gives that $\rho(Tf) < \epsilon$.

The proof of the part b) is simpler and is left as an exercise. ∎

Exercise 2.13 *Show that the map $f \to \partial f/\partial x_j$ is continuous on $\mathcal{S}(\mathbf{R}^n)$.*

Exercise 2.14 *Show that the map $f \to x_j f$ is continuous on $\mathcal{S}(\mathbf{R}^n)$.*

Finally, it would be embarrassing to discover that the space $\mathcal{S}(\mathbf{R}^n)$ contains only the zero function. The following exercise implies that $\mathcal{S}(\mathbf{R}^n)$ is not trivial.

Exercise 2.15 *a) Let*

$$\phi(t) = \begin{cases} \exp(-1/t), & t > 0 \\ 0, & t \le 0. \end{cases}$$

Show that $\phi(t)$ is in $C^\infty(\mathbf{R})$. That is, ϕ has derivatives of all orders on the real line. Hint: Show by induction that $\phi^{(k)}(t) = P_{2k}(1/t)e^{-1/t}$ for $t > 0$ where P_{2k} is a polynomial of order $2k$.

b) Show that $\phi(1 - |x|^2)$ is in $\mathcal{S}(\mathbf{R}^n)$. Hint: This is immediate from the chain rule and part a).

Lemma 2.16 *If $1 \le p < \infty$, then $\mathcal{S}(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$.*

Proof. Let ϕ be the function defined in part b) of exercise 2.15 and then define $\eta = \phi/(\int \phi\, dx)$ so that $\int_{\mathbf{R}^n} \eta(x)\, dx = 1$. Define $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$ and given f in $L^p(\mathbf{R}^n)$, set $f_\epsilon = \eta_\epsilon * f$. It is known that if $1 \le p < \infty$, then

$$\lim_{\epsilon \to 0^+} \|f_\epsilon - f\|_p = 0, \qquad 1 \le p < \infty.$$

See [40], for example.

Finally, let

$$f_{\epsilon_1,\epsilon_2}(x) = \phi(\epsilon_2 x) f_{\epsilon_1}(x).$$

Since $\phi(0) = 1$, we can choose ϵ_1 and then ϵ_2 small so that $\|f - f_{\epsilon_1,\epsilon_2}\|_p$ is as small as we like. Since $f_{\epsilon_1,\epsilon_2}$ is infinitely differentiable and compactly supported, we have proven the density of $\mathcal{S}(\mathbf{R}^n)$ in L^p. ∎

Exercise 2.17 *Show that if we take the closure of $\mathcal{S}(\mathbf{R}^n)$ in the norm of L^∞, we obtain $C_0(\mathbf{R}^n)$, the class of continuous functions on $\mathbf{R}^n$ which vanish at infinity.*

In particular, we can conclude that Lemma 2.16 does not hold for $p = \infty$.

2.2 Tempered distributions

We define the space of *tempered distributions*, $\mathcal{S}'(\mathbf{R}^n)$ as the dual of $\mathcal{S}(\mathbf{R}^n)$. If V is a topological vector space, then the *dual* is the vector space of continuous linear functionals on V. We give some examples of tempered distributions.

Example 2.18 *Each $f \in \mathcal{S}$ gives a tempered distribution by the formula*

$$g \to u_f(g) = \int_{\mathbf{R}^n} f(x)g(x)\,dx.$$

In the following example, we introduce the standard notation $\langle x \rangle = (1+|x|^2)^{1/2}$.

Example 2.19 *If f is in $L^p(\mathbf{R}^n)$ for some p, $1 \le p \le \infty$, then we may define a tempered distribution u_f by*

$$u_f(g) = \int_{\mathbf{R}^n} f(x)g(x)\,dx$$

To see this, note that if N is a non-negative integer, then $\langle x \rangle^N |f(x)|$ is bounded by a linear combination of the norms, $\rho_{\alpha 0}(f)$ for $|\alpha| \le N$. Thus, for $f \in \mathcal{S}(\mathbf{R}^n)$, we have that there exists a constant $C = C(n,N)$ so that

$$(\int |f(x)|^p\,dx)^{1/p} \le C \sum_{\alpha \le N} \rho_{\alpha 0}(f) (\int_{\mathbf{R}^n} \langle x \rangle^{-pN}\,dx)^{1/p}.$$

If we fix N so that $pN > n$, then the integral on the right-hand side of the above inequality (2.20) is finite and we obtain a constant C so that

$$\|f\|_p \le C(n,p) \sum_{|\alpha| \le N} \rho_{\alpha 0}(f). \tag{2.20}$$

Note that for $p = \infty$, the estimate $\|f\|_\infty \le \rho_{00}(f)$ is obvious. Now for f is in L^p, we have $|u_f(g)| \le \|f\|_p \|g\|_{p'}$ from Hölder's inequality. Now the inequality (2.20) applied to g and the $L^{p'}$ norm and Proposition 2.11 imply that u_f is continuous.

Exercise 2.21 *For $1 \le p \le \infty$ and $k \in \mathbf{R}$ we may define a* weighted L^p-space $L^p_k(\mathbf{R}^n) = \{f : \langle x \rangle f(x) \in L^p\}$ *with the norm*

$$\|f\|_{L^p_k} = \left(\int_{\mathbf{R}^n} |f(x)|^p \langle x \rangle^{pk}\,dx \right)^{1/p}.$$

Show that if $f \in L^p_k$ for some p and k then $u_f(g) = \int fg\,dx$ defines a tempered distribution.

Exercise 2.22 *Suppose that f is a locally integrable function and that there are constants C and N so that*

$$\int_{\{x:|x|<R\}} |f(x)|\,dx \le CR^N, \qquad R > 1.$$

Show that f defines a tempered distribution.

Exercise 2.23 *Show that the map $f \to u_f$ from $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^n)$ is injective.*

Example 2.24 *The delta function δ is the tempered distribution given by*

$$\delta(f) = f(0).$$

Example 2.25 *More generally, if μ is any finite Borel measure on $\mathbf{R}^n$, we have a distribution u_μ defined by*

$$u_\mu(f) = \int f\,d\mu.$$

This is a tempered distribution because

$$|u_\mu(f)| \le |\mu|(\mathbf{R}^n)\rho_{00}(f).$$

Example 2.26 *Any polynomial P gives a tempered distribution by*

$$u_P(f) = \int P(x)f(x)\,dx.$$

Example 2.27 *For each multi-index α, a distribution is given by*

$$\delta^{(\alpha)}(f) = \frac{\partial^\alpha f(0)}{\partial x^\alpha}.$$

2.3 Operations on tempered distributions

If T is a continuous linear map on $\mathcal{S}(\mathbf{R}^n)$ and u is a tempered distribution, then $f \to u(Tf)$ is also a distribution. The map $u \to u \circ T$ is called the transpose of T and is sometimes written as $T^t u = u \circ T$. This construction is an important part of extending familiar operations on functions to tempered distributions. Our first example considers the map

$$f \to \frac{\partial^\alpha f}{\partial x^\alpha}$$

which is clearly continuous on the Schwartz space. Thus if u is a distribution, then we can define a new distribution by

$$v(f) = u(\frac{\partial^\alpha f}{\partial x^\alpha}).$$

If we have a distribution u which is given by a Schwartz function f, we can integrate by parts and show that

$$(-1)^{|\alpha|} u_f(\frac{\partial^\alpha g}{\partial x^\alpha}) = u_{\partial^\alpha f/\partial x^\alpha}(g).$$

Thus we will define the *derivative of a distribution* u by the formula

$$\frac{\partial^\alpha u}{\partial x^\alpha}(g) = (-1)^{|\alpha|} u(\frac{\partial^\alpha g}{\partial x^\alpha}).$$

This extends the definition of derivative from smooth functions to distributions. When we say extend the definition of an operation T from functions to distributions, this means that we have

$$T u_f = u_{Tf}$$

whenever f is a Schwartz function.

This definition of the derivative on distributions is an example of a general procedure for extending maps from functions to distributions. Given a map $T : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$, we can extend T to $\mathcal{S}'(\mathbf{R}^n)$ if if we can find a (formal) transpose of T, T^t, that satisfies

$$\int_{\mathbf{R}^n} T f g \, dx = \int_{\mathbf{R}^n} f T^t g \, dx$$

for all $f, g \in \mathcal{S}(\mathbf{R}^n)$. Then if T^t is continuous on $\mathcal{S}(\mathbf{R}^n)$, we can define T on $\mathcal{S}'(\mathbf{R}^n)$ by $Tu(f) = u(T^t f)$.

Exercise 2.28 *Show that if α and β are multi-indices and u is a tempered distribution, then*

$$\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial x^\beta} u = \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\alpha}{\partial x^\alpha} u.$$

Hint: A standard result of vector calculus tells us when partial derivatives of functions commute.

Exercise 2.29 *Suppose that f is in L^p for some p with $1 \le p \le \infty$ and that the partial derivative $\partial f/\partial x_j$ exists in the L^p sense. Let u_f be the tempered distribution give by the function f and show that*

$$\frac{\partial}{\partial x_j} u_f = u_{\partial f/\partial x_j}.$$

Exercise 2.30 *Let $H(t)$ be the Heaviside function on the real line. Thus $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t < 0$. Find the distributional derivative of H. That is find $H'(\phi)$ for ϕ in $\mathcal{S}$.*

We give some additional examples of extending operations from functions to distributions. If P is a polynomial, then we $f \to Pf$ defines a continuous map on the Schwartz space. We can define multiplication of a distribution by a polynomial by $Pu(f) = u(Pf)$.

Exercise 2.31 *Show that this definition extends to ordinary product of functions in the sense that if f is a Schwartz function,*

$$u_{Pf} = Pu_f.$$

Exercise 2.32 *Show that if f and g are in $\mathcal{S}(\mathbf{R}^n)$, then fg is in $\mathcal{S}(\mathbf{R}^n)$ and that the map*

$$f \to fg$$

is continuous.

Exercise 2.33 *Show that $1/x$ defines a distribution on $\mathbf{R}$ by*

$$u(f) = \lim_{\epsilon \to 0^+} \int_{\{x:|x|>\epsilon\}} f(x)\frac{1}{x}\,dx.$$

This way of giving a value to an integral which is not defined as an absolutely convergent integral is called the principal value *of the integral. Hint: The function $1/x$ is odd, thus if we consider $\int_{\{\epsilon<|x|<1\}} f(x)/x\,dx$, we can subtract a constant from f without changing the value of the integral.*

Exercise 2.34 *Let $u(f) = \lim_{R\to\infty} \int_{-\infty}^{R} f(t)e^t\,dt$, provided the limit exists. Is u a tempered distribution?*

Exercise 2.35 *Let $u(f) = \lim_{R\to\infty} \int_{-\infty}^{R} e^t \sin(e^t)\,dt$, provided the limit exists. Is the map u a tempered distribution?*

Next we consider the convolution of a distribution and a test function. If f and g are in the Schwartz class, we have by Fubini's theorem that

$$\int_{\mathbf{R}^n} f * g(x)h(x)\,dx = \int_{\mathbf{R}^n} f(y) \int_{\mathbf{R}^n} h(x)\tilde{g}(y-x)\,dx\,dy.$$

The *reflection* of g, $\tilde{g}$ is defined by $\tilde{g}(x) = g(-x)$. Thus, we can define the convolution of a tempered distribution u and a test function g, $g * u$ by

$$g * u(f) = u(f * \tilde{g}).$$

This will be a tempered distribution thanks to exercise 2.37 below.

2.4 The Fourier transform

Proposition 2.36 *The Fourier transform is a continuous linear map from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ with a continuous inverse, $f \to \check{f}$.*

Proof. We use the criterion of Proposition 2.11 to show that the Fourier transform is continuous. If we consider the expression in a semi-norm, we have

$$\xi^\alpha \frac{\partial^\beta}{\partial \xi^\beta} \hat{f}(\xi) = (\frac{\partial^\alpha}{\partial x^\alpha} x^\beta f)\hat{}$$

where we have used Propositions 1.18 and 1.19. By the Leibniz rule in (2.4), we have

$$(\frac{\partial^\alpha}{\partial x^\alpha} x^\beta f)\hat{} = \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\delta!}((\frac{\partial^\gamma}{\partial x^\gamma} x^\beta)\frac{\partial^\delta}{\partial x^\delta} f)\hat{}.$$

Hence, using the observation of (2.20) and Proposition 1.18, we have that there is a constant $C = C(n)$ so that

$$\rho_{\alpha\beta}(\hat{f}) \le C \sum_{\lambda\le\beta, |\gamma|\le|\alpha|+n+1} \rho_{\gamma\lambda}(f).$$

Now, Proposition 2.11 implies that the map $f \to \hat{f}$ is continuous. In addition, as $\hat{f}$ is in $\mathcal{S}(\mathbf{R}^n)$ and hence in L^1, we may use the Fourier inversion theorem, Theorem 1.32 to obtain that $\check{\hat{f}} = f$.

Given (1.34) or (1.33) the continuity of $f \to \check{f}$ on $\mathcal{S}(\mathbf{R}^n)$ is immediate from the continuity of $f \to \hat{f}$ and it is clear that $\check{f}$ lies in the Schwartz space for $f \in \mathcal{S}(\mathbf{R}^n)$. Then, we can use (1.34) and then the Fourier inversion theorem for L^1, Theorem 1.32, to show

$$\hat{\check{f}} = \frac{1}{(2\pi)^n} \bar{\bar{\hat{\bar{\hat{\bar{f}}}}}} = \bar{\check{\hat{\bar{f}}}} = f.$$

∎

Exercise 2.37 *Show that if f and g are in $\mathcal{S}(\mathbf{R}^n)$, then $f * g \in \mathcal{S}(\mathbf{R}^n)$. Furthermore, show that $f \to f * g$ is continuous on $\mathcal{S}$. Hint: One way to do this is to use the Fourier transform and reduce to a problem about pointwise products.*

Next, recall the identity

$$\int \hat{f}(x)g(x)\,dx = \int f(x)\hat{g}(x)\,dx$$

of Lemma 1.31 which holds if f and g are Schwartz functions. Using this identity, it is clear that we want to define the Fourier transform of a tempered distribution by

$$\hat{u}(g) = u(\hat{g}).$$

Then the above identity implies that if u_f is a distribution given by a Schwartz function, or an L^1 function, then

$$u_{\hat{f}}(g) = \hat{u}_f(g).$$

Thus, we have defined a map which extends the Fourier transform.

In a similar way, we can define $\check{u}$ for a tempered distribution u by $\check{u}(f) = u(\check{f})$.

Theorem 2.38 *The Fourier transform is an invertible linear map on $\mathcal{S}'(\mathbf{R}^n)$.*

Proof. We know that $f \to \check{f}$ is the inverse of the map $f \to \hat{f}$ on $\mathcal{S}(\mathbf{R}^n)$. Thus, it is easy to see that $u \to \check{u}$ is an inverse to $u \to \hat{u}$ on $\mathcal{S}'(\mathbf{R}^n)$. ∎

Exercise 2.39 *If f is L^1, we have two ways to talk about the Fourier transform of f. We defined $\hat{f}$ as the Fourier transform of an L^1 function in Chapter 1 and in this chapter, we defined the Fourier transform of the distribution u_f given by $u_f(g) = \int fg\,dx$. Show that*

$$\hat{u}_f = u_{\hat{f}}.$$

Exercise 2.40 *Show that if f is in $\mathcal{S}$, then f has a derivative in the L^1-sense.*

Exercise 2.41 *Show from the definitions that if u is a tempered distribution, then*

$$(\frac{\partial^\alpha}{\partial x^\alpha}u)\hat{} = (i\xi)^\alpha \hat{u}$$

and that

$$((-ix)^\alpha u)\hat{} = (\frac{\partial^\alpha \hat{u}}{\partial \xi^\alpha}).$$

We define convergence of distributions. We will say that a *sequence of distributions* $\{u_j\}$ *converges to* u *in* $\mathcal{S}'(\mathbf{R}^n)$ if

$$\lim_{j\to\infty} u_j(f) = u(f), \quad \text{for every } f \in \mathcal{S}(\mathbf{R}^n).$$

The standard topology on the space of distributions, $\mathcal{S}'(\mathbf{R}^n)$ is the weak-* topology. To define this topology, we recall the notation u_f to denote the distribution $g \to \int_{\mathbf{R}^n} fg\,dx$. The *weak-* topology* is the weakest topology that makes the family of maps $\{u_f : f \in \mathcal{S}(\mathbf{R}^n)\}$ continuous.

Exercise 2.42 *Show that a sequence of distributions converges in the weak-* topology if and only if it converges in the sense described above.*

Exercise 2.43 *If f_j is a sequence of functions and f_j converges in $L^p(\mathbf{R}^n)$ to f, show that the distributions u_{f_j} given by f_j converge to u_f.*

Exercise 2.44 *Let u_j be a sequence of $L^1(\mathbf{R}^n)$ functions and suppose that $\hat{u}_j$ are uniformly bounded and converge pointwise a.e. to a function v. Show that we may define a tempered distribution u by $u(f) = \lim_{j\to\infty} u_j(f)$ and that $\hat{u}(f) = \int fv\,dx$.*

Exercise 2.45 *Let u be the temperered distribution defined in Exercise 2.33,*

$$u(f) = \lim_{\epsilon\to 0^+} \int_{|x|>\epsilon} f(t)\frac{dt}{t}.$$

Find $\hat{u}$. Hint: Consider the sequence of function $f_j(t) = \frac{1}{t}\chi_{\{1/j<|t|<j\}}(t)$ and use the previous exercise.

Exercise 2.46 *(Poisson summation formula) If f is in $\mathcal{S}(\mathbf{R})$, show that we have*

$$\sum_{k=-\infty}^{\infty} f(x+2\pi k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}.$$

Hint: The standard proof requires basic facts about Fourier series.

b) Show that $P = \sum_{j=-\infty}^{\infty} \delta_j$ defines a tempered distribution and that

$$\hat{P} = 2\pi \sum_{k=-\infty}^{\infty} \delta_{2\pi k}.$$

2.5 More distributions

In addition to the tempered distributions discussed above, there are two more spaces of distributions that are commonly studied. The (ordinary) distributions $\mathcal{D}'(\mathbf{R}^n)$ and the distributions of compact support, $\mathcal{E}'(\mathbf{R}^n)$. The $\mathcal{D}'$ is defined as the dual of $\mathcal{D}(\mathbf{R}^n)$, the set of functions which are infinitely differentiable and have compact support on $\mathbf{R}^n$. The space $\mathcal{E}'$ is the dual of $\mathcal{E}(\mathbf{R}^n)$, the set of functions which are infinitely differentiable on $\mathbf{R}^n$.

Since we have the containments,

$$\mathcal{D}(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n) \subset \mathcal{E}(\mathbf{R}^n),$$

we obtain the containments

$$\mathcal{E}'(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n).$$

To see this, observe that (for example) each tempered distribution defines an ordinary distribution by restricting the domain of u from $\mathcal{S}$ to $\mathcal{D}$.

The space $\mathcal{D}'(\mathbf{R}^n)$ is important because it can also be defined on open subsets of $\mathbf{R}^n$ or on manifolds. The space $\mathcal{E}'$ is interesting because the Fourier transform of such a distribution will extend holomorphically to $\mathbf{C}^n$. The books of Laurent Schwartz [26, 27] are still a good introduction to the subject of distributions.

Chapter 3

The Fourier transform on L^2.

In this section, we prove that the Fourier transform acts on L^2 and that $f \to (2\pi)^{-n/2}\hat{f}$ is an isometry on this space. Each L^2 function gives a tempered distribution and thus its Fourier transform is defined. Thus, the new result is to prove the Plancherel identity which asserts that $f \to (2\pi)^{-n/2}\hat{f}$ is an isometry.

3.1 Plancherel's theorem

Proposition 3.1 *If f and g are in the Schwartz space, then we have*

$$\int_{\mathbf{R}^n} f(x)\bar{g}(x)\,dx = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)\,d\xi.$$

Proof. According to the Fourier inversion theorem, Theorem 1.32,

$$\bar{g} = \frac{1}{(2\pi)^n}\hat{\bar{\hat{g}}}.$$

Thus, we can use the identity of Lemma 1.31 of Chapter 1 to conclude the Plancherel identity for Schwartz functions. ∎

Theorem 3.2 *(Plancherel) If f is in L^2, then $\hat{f}$ is in L^2 and we have*

$$\int |f(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\hat{f}(\xi)|^2\,d\xi.$$

Furthermore, the map $f \to \hat{f}$ is invertible.

Proof. According to Lemma 2.16 we may approximate f by a sequence of functions $\{f_j\}$ taken from the Schwartz class, $\{f_j\}$. Applying the previous proposition with $f = g = f_i - f_j$ we conclude that the sequence $\{\hat{f}_j\}$ is Cauchy in L^2. Since L^2 is complete, the sequence $\{f_j\}$ has a limit, F. Since $f_j \to F$ in L^2 we also have that $\{f_j\}$ converges to F as tempered distributions. To see this, we use the definition of the Fourier transform, and then that $\{f_j\}$ converges in L^2 to obtain that

$$u_{\hat{f}}(g) = \int f\hat{g}\,dx = \lim_{i\to\infty}\int f_i\hat{g}\,dx = \int \hat{f}_i g\,dx = \int Fg\,dx.$$

Thus $\hat{f} = F$. The identity holds for f and $\hat{f}$ since it holds for each f_j.

We know that f has an inverse on $\mathcal{S}$, $f \to \check{f}$. The Plancherel identity tells us this inverse extends continuously to all of L^2. It is easy to see that this extension is still an inverse on L^2. ∎

Recall that a *Hilbert space* $\mathcal{H}$ is a complete normed vector space where the norm comes from an inner product. An *inner product* is a map $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbf{C}$ which satisfies

$$\begin{array}{ll} \langle x,y\rangle = \overline{\langle y,x\rangle}, & \text{if } x,y\in\mathcal{H} \\ \langle \lambda x,y\rangle = \lambda\langle x,y\rangle, & x,y\in\mathcal{H},\lambda\in\mathbf{C} \\ \langle x,x\rangle \ge 0, & x\in\mathcal{H} \\ \langle x,x\rangle = 0, & \text{if and only if } x=0 \end{array}$$

Exercise 3.3 *Show that the Plancherel identity holds if f takes values in finite dimensional Hilbert space. Hint: Use a basis.*

Exercise 3.4 *Show by example that the Plancherel identity may fail if f does not take values in a Hilbert space. Hint: The characteristic function of* $(0,1) \subset \mathbf{R}$ *should provide an example. Norm the complex numbers by the ∞-norm,* $\|z\| = \max(\operatorname{Re} z, \operatorname{Im} z)$.

Exercise 3.5 *(Heisenberg inequality.) If f is a Schwartz function, show that we have the inequality:*

$$n\int_{\mathbf{R}^n} |f(x)|^2\,dx \le 2\|xf\|_2\|\nabla f\|_2.$$

Hint: Write

$$\int_{\mathbf{R}^n} n|f(x)|^2\,dx = \int_{\mathbf{R}^n} (\operatorname{div} x)|f(x)|^2\,dx$$

and integrate by parts. Recall that the gradient *operator* ∇ *and the* divergence *operator,* div *are defined by*

$$\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) \text{ and } \operatorname{div}(f_1, \dots, f_n) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}.$$

This inequality is a version of the Heisenberg uncertainty principle in quantum mechanics. The function $|f(x)|^2$ *is a probability density and thus has integral 1. The integral of* $|xf|^2$ *measures the uncertainty of the position of the particle represented by* f *and the integral of* $|\nabla f|$ *measures the uncertainty in the momentum. The inequality gives a lower bound on the product of the uncertainty in position and momentum.*

If we use Plancherel's theorem and Proposition 1.19, we obtain

$$\int_{\mathbf{R}^n} |\nabla f|^2 \, dx = (2\pi)^{-n} \int_{\mathbf{R}^n} |\xi \hat{f}(\xi)|^2 \, d\xi.$$

If we use this to replace $\|\nabla f\|_2$ *in the above inequality, we obtain a quantitative version of the statement "We cannot have* f *and* $\hat{f}$ *concentrated near the origin."*

3.2 Multiplier operators

If σ is a tempered distribution, then σ defines a *multiplier operator* $T_\sigma \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by

$$(T_\sigma f)\hat{} = \sigma \hat{f}.$$

The function σ is called the *symbol* of the operator. It is clear that T_σ maps $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$.

Our main interest is when σ is a locally integrable function. Such a function will be a tempered distribution if there are constants C and N so that

$$\int_{B_R(0)} |\sigma(\xi)| \, d\xi \leq CR^N, \text{for all } R > 1.$$

See exercise 2.22.

Exercise 3.6 *Is this condition necessary for a positive function to give a tempered distribution?*

There is a simple, but extremely useful condition for showing that a multiplier operator is bounded on L^2. Recall that in Exercise 1.3 we showed that a linear map $T : V \to W$ between normed vector spaces is continuous if and only if we have the inequality $\|Tf\|_W \leq A\|f\|_V$ for some $A < \infty$ and all $v \in V$. We introduce the *operator norm* $\|T\|_{\mathcal{L}(V,W)}$ which is defined by

$$\|T\|_{\mathcal{L}(V,W)} = \sup_{v \in V \setminus \{0\}} \frac{\|Tv\|_W}{\|v\|_V}.$$

When $V = W$, we will generally list the space once.

Theorem 3.7 *Suppose T_σ is a multiplier operator given by a measurable function m. The operator T_σ is bounded on L^2 if and only if σ is in L^∞. Furthermore, $\|T_\sigma\|_{\mathcal{L}(L^2)} = \|\sigma\|_\infty$.*

Proof. If σ is in L^∞, then Plancherel's theorem implies the inequality

$$\|T_\sigma f\|_2 \leq \|\sigma\|_\infty \|f\|_2.$$

Now consider $E_t = \{\xi : |\sigma(\xi)| > t\}$ and suppose this set has positive measure. If we choose $F_t \subset E_t$ with $0 < m(F_t) < \infty$ and so that m is bounded on F_t, then we have

$$\|T_m(\check{\chi}_{F_t})\|_2 \geq t\|\check{\chi}_{F_t}\|_2.$$

Since we have this inequality for all $t < \|\sigma\|_\infty$, we may conclude $\|T_\sigma\|_{\mathcal{L}(L^2)} \geq \|\sigma\|_\infty$. ∎

Exercise 3.8 *(Open.) Find a necessary and sufficient condition for T_m to be bounded on L^p.*

Exercise 3.9 *Fix $h \in \mathbf{R}^n$ and suppose that $\tau_h f(x) = f(x+h)$. Show that τ_h is a continuous map on $\mathcal{S}(\mathbf{R}^n)$ and find an extension of this operator to $\mathcal{S}'(\mathbf{R}^n)$.*

Exercise 3.10 *Suppose that $T_m : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$. Show that T_m commutes with translations.*

Example 3.11 *If s is a real number, then we can define J_s, the Bessel potential operator of order s by*

$$(J_s f)\hat{} = \langle \xi \rangle^{-s} \hat{f}.$$

If $s \geq 0$, then Theorem 3.7 implies that $J_s f$ lies in L^2 when f is L^2. Furthermore, if α is multi-index of length $|\alpha| \leq s$, then we have

$$\left\| \frac{\partial^\alpha}{\partial x^\alpha} J_s f \right\|_{L^2} \leq C\|f\|_2.$$

The operator $f \to \frac{\partial^\alpha}{\partial x^\alpha} J_s f$ is a multiplier operator with symbol $(i\xi)^\alpha / \langle \xi \rangle^s$, which is bounded. Since the symbol is bounded by 1, we know that this operator is bounded on L^2.

3.3 Sobolev spaces

The Example 3.11 motivates the following definition of the Sobolev space $L^{2,s}$. Sobolev spaces are so useful that each mathematician has his or her own notation for them. Some of the more common ones are H^s, $W^{s,2}$, $F_2^{2,s}$, and $B_2^{2,s}$.

Definition 3.12 *The* Sobolev space $L^{2,s}(\mathbf{R}^n)$ *is the image of* $L^2(\mathbf{R}^n)$ *under the map* J_s. *The norm is given by*

$$\|J_s f\|_{2,s} = \|f\|_2$$

or, since $J_s \circ J_{-s}$ *is the identity, we have*

$$\|f\|_{2,s} = \|J_{-s}f\|_2.$$

Note that if $s \geq 0$, then $L^2_s \subset L^2$ as observed in Example 3.11. For $s = 0$, we have $L^2_0 = L^2$. For $s < 0$, the elements of the Sobolev space are tempered distributions, which are not, in general, given by functions. The following propositions are easy consequences of the definition and the Plancherel theorem, via Theorem 3.7.

Proposition 3.13 *If* $s \geq 0$ *is an integer, then a function* f *is in the Sobolev space* $L^{2,s}$ *if and only if* f *and all its derivatives of order up to* s *are in* L^2.

Proof. If f is in the Sobolev space $L^{2,s}$, then $f = J_s \circ J_{-s} f$. Using the observation of Example 3.11 that

$$f \to \frac{\partial^\alpha}{\partial x^\alpha} J_s f$$

is bounded on L^2 when $|\alpha| \leq s$, we conclude that

$$\|\frac{\partial^\alpha}{\partial x^\alpha} f\|_2 = \|\frac{\partial^\alpha}{\partial x^\alpha} J_s \circ J_{-s} f\|_2 \leq C\|J_{-s}f\|_2 = \|f\|_{2,s}.$$

If f has all derivatives of order up to s in L^2, then we have that there is a finite constant C so that

$$\langle\xi\rangle^s |\hat{f}(\xi)| \leq C(1 + \sum_{j=1}^n |\xi_j|^s)|\hat{f}(\xi)|.$$

Since each term on the right is in L^2, we have f is in the Sobolev space, $L^{2,s}(\mathbf{R}^n)$. ∎

The characterization of Sobolev spaces in the above theorem is the more standard definition of Sobolev spaces. It is more convenient to define a Sobolev spaces for s a positive integer as the functions which have (distributional) derivatives of order less or equal s in L^2 because this definition extends easily to give Sobolev spaces on open subsets of $\mathbf{R}^n$ and Sobolev spaces based on L^p. The definition using the Fourier transform provides a nice definition of Sobolev spaces when s is not an integer.

Proposition 3.14 *If $s > 0$, $0 \leq |\alpha| \leq s$, and $f \in L^2(\mathbf{R}^n)$, then $\partial^\alpha f/\partial x^\alpha$ is in $L^{2,-s}(\mathbf{R}^n)$ and*

$$\|\frac{\partial^\alpha f}{\partial x^\alpha}\|_{2,s} \leq \|f\|_2.$$

Proof. According to Definition 3.12, it suffices to show that $J_s\frac{\partial^\alpha f}{\partial x^\alpha}$ lies in L^2. But

$$(J_s\frac{\partial^\alpha f}{\partial x^\alpha})\hat{} = \frac{(i\xi)^\alpha}{\langle x\rangle^s}\hat{f}(\xi).$$

Since $(i\xi)^\alpha/\langle\xi\rangle^s$ is bounded by one, it is easy to see the right-hand side lies in L^2 and then Theorem 3.7 implies that $J_s\frac{\partial^\alpha f}{\partial x^\alpha}$ lies in L^2. ∎

Exercise 3.15 *Show that for all s in $\mathbf{R}$, the map*

$$f \to \frac{\partial^\alpha f}{\partial x^\alpha}$$

maps $L^{2,s} \to L^{2,s-|\alpha|}$.

Exercise 3.16 *Show that $L^{2,-s}$ is the dual of $L^{2,s}$. More precisely, show that if $\lambda : L^{2,s} \to \mathbf{C}$ is a continuous linear map, then there is a distribution $u \in L^{2,-s}$ so that*

$$\lambda(f) = u(f)$$

for each $f \in \mathcal{S}(\mathbf{R}^n)$. Hint: This is an easy consequence of the theorem that all continuous linear functionals on the Hilbert space L^2 are given by $f \to \int f\bar{g}$.

Chapter 4

Interpolation of operators

In the section, we will say a few things about the theory of interpolation of operators. For a more detailed treatment, we refer the reader to the book of Stein and Weiss [31] and the book of Bergh and Löfstrom [5].

An interpolation theorem is the following type of result. If T is a linear map which is bounded [1] on X_0 and X_1, then T is bounded on X_t for t between 0 and 1. It should not be clear what we mean by "between" when we are talking about pairs of vector spaces. In the context of L^p spaces, L^q is between L^p and L^r will mean that q is between p and r.

For these results, we will work on a pair of σ-finite measure spaces $(M, \mathcal{M}, \mu)$ and $(N, \mathcal{N}, \nu)$.

4.1 The Riesz-Thorin theorem

We begin with the Riesz-Thorin convexity theorem.

Theorem 4.1 *Let p_j, q_j, $j = 0, 1$ be exponents in the range $[1, \infty]$. If T is a linear operator defined (at least) on simple functions in $L^1(M)$ into measurable functions on N that satisfies*

$$\|Tf\|_{q_j} \leq A_j \|f\|_{p_j}.$$

If we define p_t and q_t by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1},$$

[1] A linear map $T : X \to Y$ is bounded as a map from the normed vector space X to Y if the inequality $\|Tf\|_Y \leq C\|f\|_X$ holds. The least constant C for which this inequality holds is called the operator norm of T.

then we have that

$$\|Tf\|_{q_t} \leq A_t \|f\|_{p_t}.$$

The operator norm, A_t satisfies $A_t \leq A_0^{1-t} A_1^t$. If $p_t < \infty$, then we have that T extends to a bounded operator operator $T : L^{p_t}(M) \to L^{q_t}(N)$.

Before giving the proof of the Riesz-Thorin theorem, we look at some applications.

Proposition 4.2 *(Hausdorff-Young inequality) The Fourier transform satisfies for $1 \leq p \leq 2$*

$$\|\hat{f}\|_{p'} \leq (2\pi)^{n/p'} \|f\|_p.$$

Proof. Proposition 1.2 tells us that the Fourier transform maps L^1 to L^∞ and Plancherel's theorem, Theorem 3.2 tells us that the Fourier transform maps L^2 into itself. Using the Riesz-Thorin Theorem, Theorem 4.1 gives that the Fourier transform satisfies

$$\|\hat{f}\|_{q_t} \leq (2\pi)^{tn/2} \|f\|_{p_t}$$

with $1/p_t = (1-t) + t/2$ and $1/q_t = t/2$, $0 \leq t \leq 1$. If we express t and q_t in terms of p_t to obtain $t = 2/p'_t$ and $1/q_t = 1/p'_t$ to give the desired result. ∎

The next result appeared as an exercise when we introduced convolution.

Proposition 4.3 *(Young's convolution inequality) If $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$, $1 \leq p, q, r \leq \infty$ and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

then

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. We fix p, $1 \leq p \leq \infty$, and let $f \in L^p(\mathbf{R}^n)$. We will apply Theorem 4.1 to the map $g \to f * g$. Our endpoints are Hölder's inequality which gives

$$|f * g(x)| \leq \|f\|_p \|g\|_{p'}$$

and thus $g \to f * g$ maps $L^{p'}(\mathbf{R}^n)$ to $L^\infty(\mathbf{R}^n)$ and the simpler version of Young's inequality which tells us that if g is in L^1, then

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Thus $g \to f * g$ also maps L^1 to L^p. Thus, this map also takes L^{q_t} to L^{r_t} where

$$\frac{1}{q_t} = \frac{1-t}{1} + t(1 - \frac{1}{p}) \quad \text{and} \quad \frac{1}{r_t} = \frac{1-t}{p} + \frac{t}{\infty}.$$

If we subtract the definitions of $1/r_t$ and $1/q_t$, then we obtain the relation

$$\frac{1}{r_t} - \frac{1}{q_t} = 1 - \frac{1}{p}.$$

The condition $q \geq 1$ is equivalent with $t \geq 0$ and $r \geq 1$ is equivalent with the condition $t \leq 1$. Thus, we obtain the stated inequality for precisely the exponents p, q and r in the hypothesis. ∎

Exercise 4.4 *The simple version of Young's inequality used in the proof above can be proven directly using Hölder's inequality. A proof can also be given which uses the Riesz-Thorin theorem. To do this, use Tonelli's and then Fubini's theorem to establish the inequality*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

The other endpoint is Hölder's inequality:

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_\infty.$$

Then, apply Theorem 4.1 to the map $g \to f * g$.

Below is a simple, useful result that is a small generalization of the simple version of Young's inequality.

Exercise 4.5 *a) Suppose that $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ is measurable and that*

$$\int_{\mathbf{R}^n} |K(x,y)|\, dy \leq M_\infty$$

and

$$\int_{\mathbf{R}^n} |K(x,y)|\, dx \leq M_1.$$

Show that

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y) f(y)\, dy$$

defines a bounded operator T on L^p and

$$\|Tf\|_p \leq M_1^{1/p} M_\infty^{1/p'} \|f\|_p.$$

Hint: Show that M_1 is an upper bound for the operator norm on L^1 and M_∞ is an upper bound for the operator norm on L^∞ and then interpolate with the Riesz-Thorin Theorem, Theorem 4.1.

b) Use the result of part a) to provide a proof of Young's convolution inequality

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

To do this, write $f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy$ and then let $K(x,y) = f(x-y)$.

Our next step is a lemma from complex analysis, that is called the three lines theorem. This is one of a family of theorems which state that the maximum modulus theorem continues to hold in unbounded regions, provided we put an extra growth condition at infinity. This theorem considers analytic functions in the strip $\{z : a \leq \operatorname{Re} z \leq b\}$.

Lemma 4.6 *(Three lines lemma) If f is analytic in the strip $\{z : a < \operatorname{Re} z < b\}$ and f is bounded and continuous in the strip $\{z : a \leq \operatorname{Re} z \leq b\}$. Define*

$$M_a = \sup |f(a+it)| \qquad \textit{and} \qquad M_b = \sup |f(b+it)|,$$

then

$$|f(x+iy)| \leq M_a^{\frac{b-x}{b-a}} M_b^{\frac{x-a}{b-a}}.$$

Proof. We consider $f_\epsilon(x+iy) = e^{\epsilon(x+iy)^2} f(x+iy) M_a^{\frac{x+iy-b}{b-a}} M_b^{\frac{a-(x+iy)}{b-a}}$ for $\epsilon > 0$. This function satisfies

$$|f_\epsilon(a+iy)| \leq e^{\epsilon a^2} \qquad \text{and} \qquad |f_\epsilon(b+iy)| \leq e^{\epsilon b^2}.$$

and

$$\lim_{y \to \pm\infty} \sup_{a \leq x \leq b} |f_\epsilon(x+iy)| = 0.$$

Thus by applying the maximum modulus theorem on sufficiently large rectangles, we can conclude that for each $z \in S$,

$$|f_\epsilon(z)| \leq \max(e^{\epsilon a^2}, e^{\epsilon b^2}).$$

Letting $\epsilon \to 0^+$ implies the Lemma. ∎

Exercise 4.7 *If instead of assuming that f is bounded, we assume that*

$$|f(x+iy)| \leq e^{M|y|}$$

for some $M > 0$, then the above Lemma holds and with the same proof. Show this. What is the best possible growth condition for which the above proof works? What is the best possible growth condition? See [28].

The proof of the Riesz-Thorin theorem relies on constructing the following family of simple functions.

Lemma 4.8 *Let p_0, p_1 and p with $p_0 < p < p_1$ be given. Consider $s = \sum_{j=1}^J \alpha_j a_j \chi_{E_j}$ be a simple function with α_j are complex numbers of length 1, $|\alpha_j| = 1$, $a_j > 0$ and $\{E_j\}$ is a pairwise disjoint collection of measurable sets where each is of finite measure. Suppose $\|s\|_p = 1$. Let*

$$\frac{1}{p_z} = \frac{1-z}{p_0} + \frac{z}{p_1}$$

and define

$$s_z = \sum_{j=1}^{J} \alpha_j a_j^{p/p_z} \chi_{E_j}.$$

This family satisfies

$$\|s_z\|_{p_{\mathrm{Re}\, z}} = 1, \quad \text{for } 0 \leq \mathrm{Re}\, z \leq 1.$$

Proof. We have that

$$\int_M |s_z|^{p_{\mathrm{Re}\, z}} \, d\mu = \sum_{j=1}^{J} a_j^p \mu(E_j).$$

∎

Exercise 4.9 *State and prove a similar lemma for the family of Sobolev spaces. Suppose that u lies in $L^2_s(\mathbf{R}^n)$ with $s_0 < s < s_1$ and $\|u\|_{L^2_s} = 1$. Let $s_t = (1-t)s_0 + ts_1$ and find a family of distributions u_z so that*

$$\|u_z\|_{L^2_{s_{\mathrm{Re}\, z}}} = 1, \qquad 0 \leq t \leq 1$$

This family will be analytic in the sense that if $f \in \mathcal{S}(\mathbf{R}^n)$, then $u_z(f)$ is analytic.

We are now ready to give the proof of the Riesz-Thorin theorem, Theorem 4.1.

Proof of Riesz-Thorin theorem. We are now ready to give the proof of the Riesz-Thorin theorem, Theorem 4.1. We fix a $p = p_{t_0}$, $0 < t_0 < 1$ and consider simple functions s on M and s' on N which satisfy $\|s\|_{p_{t_0}} = 1$ and $\|s'\|_{q'_{t_0}} = 1$. In the case that $p_{t_0} < \infty$ and $q_{t_0} < \infty$, we may let s_z and s'_z be the families from Lemma 4.8 where s_z is constructed using p_j, $j = 0, 1$ and s'_z is constructed using the exponents q'_j, $j = 0, 1$.

According to our hypothesis,

$$\phi(z) = \int_N s'_z(x) T s_z(x)\, d\nu(x)$$

is an analytic function of z. Also, using Lemma 4.8 and the assumption on T,

$$\sup_{y \in \mathbf{R}} |\phi(j + iy)| \le A_j, \qquad j = 0, 1.$$

Thus by the three lines theorem, Lemma 4.6, we can conclude that

$$|\int s' T s\, d\mu| \le A_0^{1-t_0} A_1^{t_0}.$$

Since, s' is an arbitrary simple function with norm 1 in $L^{q'}$, we can conclude that

$$\|Ts\|_{q_{t_0}} \le A_0^{1-t_0} A_1^{t_0}.$$

Finally, since simple functions are dense in L^{p_t}, we may take a limit to conclude that T can be extended to all of L^p and is bounded.

In the case that $p_{t_0} = \infty$, then we have that $p_0 = p_1 = \infty$. If $q'_{t_0} < \infty$, we may still use the above proof with s_z constant to conclude that T maps into $L^{q_{t_0}}$. If $q_0 = q_1 = q_{t_0} = 1$ and $p_0 < p_1$, we may again adapt the above proof to show T maps $L^{p_{t_0}}$ into L^p.

As long as $p_t < \infty$, simple functions are dense in L^{p_t} and it is a standard fact that we may extend T to all of L^{p_t}. ∎

The next exercise may be used to carry the extension of T from simple functions to all of L^p.

Exercise 4.10 *Suppose $T : A \to Y$ is a map defined on a subset A of a metric space X into a metric space Y. Show that if T is uniformly continuous, then T has a unique continuous extension $\bar{T} : \bar{A} \to Y$ to the closure of A, $\bar{A}$. If in addition, X is a vector space, A is a subspace and T is linear, then the extension is also linear.*

Exercise 4.11 *Show that if T is a linear map (say defined on $\mathcal{S}(\mathbf{R}^n)$) which maps $L^2_{s_j}$ into $L^2_{r_j}$ for $j = 0, 1$, then T maps $L^2_{s_t}$ into $L^2_{r_t}$ for $0 < t < 1$, where $s_t = (1-t)s_0 + ts_1$ and $r_t = (1-t)r_0 + tr_1$.*

4.2 Interpolation for analytic families of operators

The main point of this section is that in the Riesz-Thorin theorem, we might as well let the operator T depend on z. This is a very simple idea which has many clever applications.

I do not wish to get involved the technicalities of analytic operator valued functions. (And am not even sure if there are any technicalities needed here.) If one examines the above proof, we see that the hypothesis we will need on an operator T_z is that for all sets of finite measure, $E \subset M$ and $F \subset N$, we have that

$$z \to \int_N \chi_E T_z(\chi_F)\, d\nu \tag{4.12}$$

is an analytic function of z. This hypothesis can often be proven by using Morera's theorem which replaces the problem of determining analyticity by the simpler problem of checking if an integral condition holds. The integral condition can often be checked with Fubini's theorem.

Theorem 4.13 *(Stein's interpolation theorem) For z in $S = \{z : 0 \le \operatorname{Re} z \le 1\}$, let T_z be a family of linear operators defined simple functions for which we have that the function in 4.12 is bounded and analytic in S. We assume that for $j = 0, 1$, T_{j+iy} maps $L^{p_j}(M)$ to $L^{q_j}(N)$. Also assume that $1 \le p_0 < p_1 \le \infty$. We let p_t and q_t have the meanings as in the Riesz-Thorin theorem and define*

$$M_t = \sup_{y \in \mathbf{R}} \|T_{t+iy}\|$$

where $\|T_{t+iy}\|$ denotes the norm of T_{t+iy} as an operator from $L^{p_t}(M)$ to $L^{q_t}(N)$. We conclude that T_t maps L^{p_t} to L^{q_t} and we have

$$M_t \le M_0^{1-t} M_1^t.$$

The proof of this theorem is the same as the proof of the Riesz-Thorin theorem.

Exercise 4.14 *(Interpolation with change of measure) Suppose that T is a linear map which which maps $L^{p_j}(d\mu)$ into $L^{q_j}(\omega_j d\nu)$ for $j = 0, 1$. Suppose that ω_0 and ω_1 are two non-negative functions which are integrable on every set of finite measure in N. Show that T maps $L^{p_t}(d\mu)$ into $L^{q_t}(\omega_t)$ for $0 < t < 1$. Here, q_t and p_t are defined as in the Riesz-Thorin theorem and $\omega_t = \omega_0^{1-t}\omega_1^t$.*

Exercise 4.15 *Formulate and prove a similar theorem where both measures μ and ν are allowed to vary.*

4.3 Real methods

In this section, we give a special case of the Marcinkiewicz interpolation theorem. This is a special case because we assume that the exponents $p_j = q_j$ are the same. The full theorem includes the off-diagonal case which is only true when $q \geq p$. To indicate the idea of the proof, suppose that we have a map T which is bounded on L^{p_0} and L^{p_1}. If we take a function f in L^p, with p between $p_0 < p < p_1$, then we may truncate f by setting

$$f_\lambda = \begin{cases} f, & |f| \leq \lambda \\ 0, & |f| > \lambda. \end{cases} \tag{4.16}$$

and then $f^\lambda = f - f_\lambda$. Since f^λ is L^{p_0} and f_λ is in L^{p_1}, then we can conclude that $Tf = Tf_\lambda + Tf^\lambda$ is defined. As we shall see, if we are clever, we can do this splitting in such a way to see that not only is Tf defined, but we can also compute the norm of Tf in L^p. The theorem applies to operators which are more general than bounded linear operators. Instead of requiring the operator T to be bounded, we require the following condition. Let $0 < q \leq \infty$ and $0 < p < \infty$ we say that T is *weak-type* p, q if there exists a constant A so that

$$\mu(\{x : |Tf(x)| > \lambda\}) \leq \left(\frac{A\|f\|_p}{\lambda} \right)^q.$$

If $q = \infty$, then an operator is of *weak-type* p, ∞ if there exists a constant A so that

$$\|Tf\|_\infty \leq A\|f\|_p.$$

We say that a map T is *strong-type* p, q if there is a constant A so that

$$\|Tf\|_q \leq A\|f\|_p.$$

For linear operators, this condition is the same as boundedness. The justification for introducing the new term "strong-type" is that we are not requiring the operator T to be linear.

Exercise 4.17 *Show that if T is of strong-type p, q, then T is of weak-type p, q. Hint: Use Chebyshev's inequality.*

The condition that T is linear is replaced by the condition that T is *sub-linear*. This means that for f and g in the domain of T, then

$$|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

The proof of the main theorem will rely on the following well-known representation of the L^p norm of f.

Lemma 4.18 *Let $p < \infty$ and f be measurable, then*

$$\|f\|_p^p = p\int_0^\infty \mu(\{x : |f(x)| > \lambda\})\lambda^p \frac{d\lambda}{\lambda}.$$

Proof. It is easy to see that this holds for simple functions. Write a general function as an increasing limit of simple functions. ∎

Our main result is the following theorem:

Theorem 4.19 *Let $0 < p_0 < p_1 \leq \infty$ for $j = 0, 1$ and let T take measurable functions on M to measurable functions on N. Assume also that T is sublinear and the domain of T is closed under taking truncations. If T is of weak-type p_j, p_j for $j = 0, 1$, then T is of strong-type p_t, p_t for $0 < t < 1$ and we have for $p_0 < p < p_1$, that when $p_1 < \infty$*

$$\|Tf\|_p \leq 2(\frac{pA_0^{p_0}}{p-p_0} + \frac{p_1A_1^{p_1}}{p_1-p})^{1/p}\|f\|_p.$$

When $p_1 = \infty$, we obtain

$$\|Tf\|^p \leq (1 + A_1)(\frac{A_0^{p_0}p}{p-p_0})^{1/p}\|f\|_p^p.$$

Proof. We first consider the case when $p_1 < \infty$ We fix $p = p_t$ with $0 < t < 1$, choose f in the domain of T and let $\lambda > 0$. We write $f = f_\lambda + f^\lambda$ as in (4.16). Since T is sub-linear and then weak-type p_j, p_j, we have that

$$\begin{aligned} \nu(\{x : |Tf(x)| > 2\lambda\}) &\leq \nu(\{x : |Tf^\lambda(x)| > \lambda\}) + \nu(\{x : |Tf_\lambda(x)| > \lambda\}) \\ &\leq \left(\frac{A_0\|f^\lambda\|_{p_0}}{\lambda}\right)^{p_0} + \left(\frac{A_1\|f_\lambda\|_{p_1}}{\lambda}\right)^{p_1}. \qquad (4.20) \end{aligned}$$

We use the representation of the L^p-norm in Lemma 4.18, the inequality (4.20) and the change of variables $2\lambda \to \lambda$ to obtain

$$\begin{aligned} 2^{-p}\|Tf\|_p^p \leq\ & A_0^{p_0}pp_0\int_0^\infty\int_0^\infty \mu(\{x : |f^\lambda(x)| > \tau\})\tau^{p_0}\frac{d\tau}{\tau}\lambda^{p-p_0}\frac{d\lambda}{\lambda} \\ &+A_1^{p_1}pp_1\int_0^\infty\int_0^\lambda \mu(\{x : |f_\lambda(x)| > \tau\})\tau^{p_1}\frac{d\tau}{\tau}\lambda^{p-p_1}\frac{d\lambda}{\lambda}. \qquad (4.21) \end{aligned}$$

Note that the second integral on the right extends only to λ since f_λ satisfies the inequality $|f_\lambda| \leq \lambda$. We consider the second term first. We use that $\mu(\{x : |f_\lambda(x)| > \tau\}) \leq \mu(\{x :$

$|f(x)| > \tau\})$ and thus Tonelli's theorem gives the integral in the second term is bounded by

$$\begin{aligned} pp_1 \int_0^\infty \mu(\{x : |f_\lambda(x)| > \tau\}) \int_\tau^\infty \lambda^{p-p_1} \frac{d\lambda}{\lambda}\frac{d\tau}{\tau} &\leq \frac{pp_1}{(p_1-p)} \int_0^\infty \mu(\{x : |f(x)| > \tau\})\tau^p \frac{d\tau}{\tau} \\ &= \frac{p_1}{p-p_1}\|f\|_p^p. \end{aligned} \tag{4.22}$$

We now consider the first term to the right of the inequality sign in (4.21). We observe that when $\tau > \lambda$, $\mu(\{x : |f^\lambda(x)| > \tau\}) = \mu(\{x : |f(x)| > \tau\})$, while when $\tau \leq \lambda$, we have $\mu(\{x : |f^\lambda(x)| > \tau\}) = \mu(\{x : |f(x)| > \lambda\})$. Thus, we have

$$\begin{aligned} & pp_0 \int_0^\infty \int_0^\infty \mu(\{x : |f^\lambda(x)| > \tau\})\tau^{p_0}\frac{d\tau}{\tau}\lambda^{p-p_1}\frac{d\lambda}{\lambda} \\ &= pp_0 \int_0^\infty \int_\lambda^\infty \mu(\{x : |f(x)| > \tau\})\tau^{p_0}\frac{d\tau}{\tau}\lambda^{p-p_0}\frac{d\lambda}{\lambda} \\ &\qquad + p\int_0^\infty \mu(\{x : |f(x)| > \lambda\})\lambda^p\frac{d\lambda}{\lambda} \\ &= (\frac{p_0}{p-p_0} + 1)\|f\|_p^p. \end{aligned} \tag{4.23}$$

Using the estimates (4.22) and (4.23) in (4.21) gives

$$\|Tf\|_p^p \leq 2^p(\frac{pA_0^{p_0}}{p-p_0} + \frac{p_1A_1^{p_1}}{p_1-p})\|f\|_p^p.$$

Which is what we hoped to prove.

Finally, we consider the case $p_1 = \infty$. Since T is of type ∞, ∞, then we can conclude that, with f_λ as above, $\nu(\{x : |Tf_\lambda(x)| > A_1\lambda\}) = 0$. To see how to use this, we write

$$\begin{aligned} \nu(\{x : |Tf(x)| > (1+A_1)\lambda\}) &\leq \nu(\{x : |Tf^\lambda(x)| > \lambda\}) + \nu(\{x : |Tf_\lambda(x)| > A_1\lambda\}) \\ &= \nu(\{x : |Tf^\lambda(x)| > \lambda\}). \end{aligned}$$

Thus, using Lemma 4.18 that T is of weak-type p_0, p_0, and the calculation in (4.22) we have

$$\begin{aligned} (1+A_1)^{-p}\|Tf\|_p^p &= A_0^{p_0}pp_0 \int_0^\infty \int_\lambda^\infty \mu(\{x : |f^\lambda(x)| > \tau\})\tau^{p_0}\frac{d\tau}{\tau}\lambda^{p-p_0}\frac{d\lambda}{\lambda} \\ &\leq \frac{A_0^{p_0}p}{p-p_0}\|f\|_p^p. \end{aligned}$$

∎

Chapter 5

The Hardy-Littlewood maximal function

In this chapter, we introduce the Hardy-Littlewood maximal function and prove the Lebesgue differentiation theorem. This is the missing step of the Fourier uniqueness theorem in Chapter 1.

Since the material in this chapter is familiar from real analysis, we will omit some of the details. In this chapter, we will work on $\mathbf{R}^n$ with Lebesgue measure.

5.1 The L^p-inequalities

We let $\chi = n\chi_{B_1(0)}/\omega_{n-1}$ be the characteristic function of the unit ball, normalized so that $\int \chi\, dx = 1$ and then we set $\chi_r(x) = r^{-n}\chi(x/r)$. If f is a measurable function, we define the *Hardy-Littlewood maximal function* by

$$Mf(x) = \sup_{r>0} |f| * \chi_r(x).$$

Here and throughout these notes, we use $m(E)$ to denote the *Lebesgue measure* of a set E.

Note that the Hardy-Littlewood maximal function is defined as the supremum of an uncountable family of functions. Thus, the sort of person who is compulsive about details might worry that Mf may not be measurable. The following lemma implies the measurability of Mf.

Lemma 5.1 *If f is measurable, then Mf is lower semi-continuous.*

Recall that an extended real-valued function $f : \mathbf{R}^n \to [-\infty, \infty]$ is *lower semi-continuous* if and only if the sets $\{x : f(x) > \lambda\}$ are open for all λ.

Proof. If $Mf(x) > \lambda$, then we can find a radius r so that

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)|\, dy > \lambda.$$

Since this inequality is strict, for s slightly larger than r, say $r + \delta > s > r$, we still have

$$\frac{1}{m(B_s(x))} \int_{B_r(x)} |f(y)|\, dy > \lambda.$$

But then by the monotonicity of the integral,

$$Mf(z) > \lambda$$

if $B_s(z) \supset B_r(x)$. That is if $|z - x| < \delta$. We have shown that the set $\{x : Mf(x) > \lambda\}$ is open. ∎

Exercise 5.2 *If $\{f_\alpha : \alpha \in I\}$ is a family of continuous real valued functions on $\mathbf{R}^n$ show that*

$$g(x) = \sup_{\alpha \in I} f_\alpha(x)$$

is lower semi-continuous.

If f is locally integrable, then $\chi_r * f$ is continuous for each $r > 0$ and the previous exercise can be used to establish the lower semi-continuity of Mf. Our previous lemma also applies to functions for which the integral over a ball may be infinite.

We pause briefly to define local integrability. We say that a function is *locally integrable* if it is in $L^1_{loc}(\mathbf{R}^n)$. We say that a function f is $L^p_{loc}(\mathbf{R}^n)$ if $f \in L^p(K)$ for each compact set K. One may also define a topology by defining the semi-norms,

$$\rho_n(f) = \|f\|_{L^p(B_n(0))}, \qquad \text{for } n = 1, 2 \ldots.$$

Given this countable family of semi-norms it is easy to define a topology by following the procedure we used to define a topology on the Schwartz space (see exercise 2.10).

Exercise 5.3 *Show that a sequence converges in the metric for $L^p_{loc}(\mathbf{R}^n)$ if and only if the sequence converges in $L^p(K)$ for each compact set K.*

Exercise 5.4 *Let $f = \chi_{(-1,1)}$ on the real line. Show that $Mf \geq 1/|x|$ if $|x| > 1$. Conclude that Mf is not in L^1.*

Exercise 5.5 *Let $f(x) = \chi_{((0,1/2))}(x)x^{-1}(\log(x))^{-2}$. Show that there is a constant $c > 0$ so that*

$$Mf(x) \geq \frac{c}{x(-\log(x))}, \qquad \text{if } 0 < x < 1/2.$$

Remark: This exercise shows that there is a function f in L^1 for which f^* is not locally integrable.

Exercise 5.6 *Show that if Mf is in $L^1(\mathbf{R}^n)$, then f is zero.*

Exercise 5.7 *Let $\{E_\alpha\}_{\alpha\in I}$ be a collection of measurable sets. Suppose there is a constant c_0 so that for each α we may find a ball $B_r(0)$ so that $E_\alpha \subset B_r(0)$ and so that $m(E_\alpha) \geq c_0 m(B_r(0))$. Define a maximal function by*

$$Nf(x) = \sup\{\frac{1}{m(E_\alpha)}\int_{E_\alpha} |f(x+y)|\, dy : \alpha \in I\}.$$

a) Show that there is a constant so that

$$Nf(x) \leq CMf(x).$$

b) Show that if $\{E_\alpha\}$ is the collection of all cubes containing 0, then we may find a constant C depending only on dimension so that

$$C^{-1}Nf(x) \leq Mf(x) \leq CMf(x).$$

We will show that the Hardy-Littlewood maximal function is finite a.e. when f is in $L^1(\mathbf{R}^n)$. This is one consequence of the following theorem.

Theorem 5.8 *If f is measurable and $\lambda > 0$, then there exists a constant $C = C(n)$ so that*

$$m(\{x : |Mf(x)| > \lambda\}) \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} |f(x)|\, dx.$$

The observant reader will realize that this theorem asserts that the Hardy-Littlewood maximal operator is of weak-type $1, 1$. It is easy to see that it is sub-linear and of weak type ∞, ∞ and thus by the diagonal case of the Marcinkiewicz interpolation theorem, Theorem 4.19, we can conclude the maximal operator is of strong-type p, p.

The proof of this theorem depends on a Lemma which allows us to extract from a collection of balls, a sub-collection whose elements are disjoint and whose total measure is relatively large.

Lemma 5.9 *Let $\beta = 1/(2\cdot 3^n)$. If E is a measurable set of finite measure in $\mathbf{R}^n$ and we have a collection of balls $\mathcal{B} = \{B_\alpha\}_{\alpha\in A}$ so that $E \subset \cup B_\alpha$, then we can find a sub-collection of the balls $\{B_1,\dots,B_N\}$ which are pairwise disjoint and which satisfy*

$$\sum_{j=1}^{N} m(B_j) \geq \beta m(E).$$

Proof. We may find $K \subset E$ which is compact and with $m(K) > m(E)/2$. Since K is compact, there is a finite sub-collection of the balls $\mathcal{B}_1 \subset \mathcal{B}$ which cover E. We let B_1 be the largest ball in $\mathcal{B}_1$ and then we let $\mathcal{B}_2$ be the balls in $\mathcal{B}_1$ which do not intersect B_1. We choose B_2 to be the largest ball in $\mathcal{B}_2$ and continue until $\mathcal{B}_{N+1}$ is empty. The balls $B_1, B_2, \dots, B_N$ are disjoint by construction. If B is a ball in $\mathcal{B}_1$ then either B is one of the chosen balls, call it B_{j_0} or B was discarded in going from $\mathcal{B}_{j_0}$ to $\mathcal{B}_{j_0+1}$ for some j_0. In either case, B intersects one of the chosen balls, B_{j_0}, and B has radius which is less than or equal to the radius of B_{j_0}. Hence, we know that

$$K \subset \cup_{B\in\mathcal{B}_1} B \subset \cup_{j=1}^{N} 3B_j$$

where if $B_j = B_r(x)$, then $3B_j = B_{3r}(x)$. Taking the measure of the sets K and $\cup 3B_j$, we obtain

$$m(E) \leq 2m(K) \leq 2\cdot 3^n \sum_{j=1}^{N} m(B_j).$$

∎

Now, we can give the proof of the weak-type 1,1 estimate for Mf in Theorem 5.8.

Proof. (*Proof of Theorem 5.8)* We let $E_\lambda = \{x : Mf(x) > \lambda\}$ and choose a measurable set $E \subset E_\lambda$ which is of finite measure. For each $x \in E_\lambda$, there is a ball B_x so that

$$m(B_x)^{-1} \int_{B_x} |f(x)|\,dx > \lambda \tag{5.10}$$

We apply Lemma 5.9 to the collection of balls $\mathcal{B} \subset \{B_x : x \in E\}$ to find a sub-collection $\{B_1,\dots,B_N\} \subset \mathcal{B}$ of disjoint balls so that

$$\frac{m(E)}{2\cdot 3^n} \leq \sum_{j=1}^{N} m(B_j) \leq \frac{1}{\lambda}\int_{B_j} |f(y)|\,dy \leq \frac{\|f\|_1}{\lambda}.$$

The first inequality above is part of Lemma 5.9, the second is (5.10) and the last holds because the balls B_j are disjoint. Since E is an arbitrary, measurable subset of E_λ of finite measure, then we can take the supremum over all such E and conclude E_λ also satisfies

$$m(E_\lambda) \leq \frac{2 \cdot 3^n \|f\|_1}{\lambda}.$$

∎

Frequently, in analysis it becomes burdensome to keep track of the exact value of the constant C appearing in the inequality. In the next theorem and throughout these notes, we will give the constant and the parameters it depends on without computing its exact value. In the course of a proof, the value of a constant C may change from one occurrence to the next. Thus, the expression $C = 2C$ is true even if $C \neq 0$!

Theorem 5.11 *If f is measurable and $1 < p \leq \infty$, then there exists a constant $C = C(n)$*

$$\|Mf\|_p \leq \frac{Cp}{p-1}\|f\|_p.$$

Proof. This follows from the weak-type 1,1 estimate in Theorem 5.8, the elementary inequality that $\|Mf\|_\infty \leq \|f\|_\infty$ and Theorem 4.19. The dependence of the constant can be read off from the constant in Theorem 4.19. ∎

5.2 Differentiation theorems

The Hardy-Littlewood maximal function is a gadget which can be used to study the identity operator. At first, this may sound like a silly thing to do–what could be easier to understand than the identity? We will illustrate that the identity operator can be interesting by using the Hardy-Littlewood maximal function to prove the Lebesgue differentiation theorem–the identity operator is a pointwise limit of averages on balls. In fact, we will prove a more general result which was used in the proof of the Fourier inversion theorem of Chapter 1. This theorem amounts to a complicated representation of the identity operator. In addition, we will introduce approximations of the zero operator, $f \to 0$ in a few chapters.

The maximal function is constructed by averaging using balls, however, it is not hard to see that many radially symmetric averaging processes can be estimated using M. The following useful result is lifted from Stein's book [29]. Before stating this proposition, given a function ϕ on $\mathbf{R}^n$, we define the non-increasing radial majorant of ϕ by

$$\phi^*(x) = \sup_{|y|>|x|} |\phi(y)|.$$

Proposition 5.12 *Let ϕ be in L^1 and f in L^p, then*

$$\sup_{r>0} |\phi_r * f(x)| \leq \int \phi^*(x)\, dx Mf(x).$$

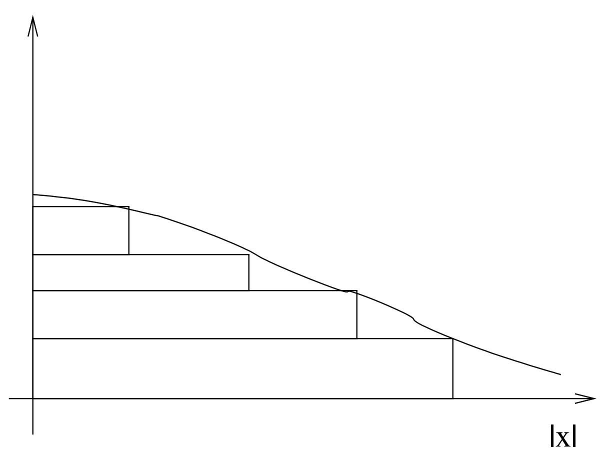
Proof. It suffices to prove the inequality

$$\phi_r * f(x) \leq \int \phi(x)\, dx Mf(x)$$

when ϕ is non-negative and radially non-increasing and thus $\phi = \phi^*$ a.e. Also, we may assume $f \geq 0$. We begin with the special case when $\phi(x) = \sum_j a_j \chi_{B_{\rho_j}(0)}(x)$ and then

$$\begin{aligned} \phi_r * f(x) &= r^{-n} \sum_j a_j \frac{m(B_{r\rho_j}(x))}{m(B_{r\rho_j}(x))} \int_{B_{r\rho_j}(x)} f(y)\, dy \\ &\leq r^{-n} Mf(x) \sum_j a_j m(B_{r\rho_j}(x)) \\ &= Mf(x) \int \phi, \qquad a.e.. \end{aligned}$$

The remainder of the proof is a picture. We can write a general, non-increasing, radial function as an increasing limit of sums of characteristic functions of balls. The monotone convergence theorem and the special case already treated imply that $\phi_r * f(x) \leq Mf(x) \int \phi\, dx$ and the Proposition follows. ∎



Finally, we give the result that is needed in the proof of the Fourier inversion theorem. We begin with a Lemma. Note that this Lemma suffices to prove the Fourier inversion theorem in the class of Schwartz functions. The full differentiation theorem is only needed when f is in L^1.

Lemma 5.13 *If f is continuous and bounded on $\mathbf{R}^n$ and $\phi \in L^1(\mathbf{R}^n)$, then for all x,*

$$\lim_{\epsilon \to 0^+} \phi_\epsilon * f(x) = f(x) \int \phi.$$

Proof. Fix x in $\mathbf{R}^n$ and $\eta > 0$. Recall that $\int \phi_\epsilon$ is independent of ϵ and thus we have

$$\phi_\epsilon * f(x) - f(x) \int \phi(x)\, dx = \int \phi_\epsilon(y)(f(x-y) - f(x))\, dy$$

Since f is continuous at x, there exists $\delta > 0$ so that $|f(x-y) - f(x)| < \eta$ if $|y| < \delta$. In the last integral above, we consider $|y| < \delta$ and $|y| \geq \delta$ separately. We use the continuity of f when $|y|$ is small and the boundedness of $|f|$ for $|y|$ large to obtain:

$$|\phi_\epsilon * f(x) - f(x) \int \phi\, dx| \leq \eta \int_{\{y:|y|<\delta\}} |\phi_\epsilon(y)|\, dy + 2\|f\|_\infty \int_{\{y:|y|>\delta\}} |\phi_\epsilon(y)|\, dy$$

The first term on the right is finite since ϕ is in L^1 and in the second term, a change of variables and the dominated convergence theorem implies we have

$$\lim_{\epsilon \to 0^+} \int_{\{y:|y|>\delta\}} |\phi_\epsilon(y)|\, dy = \lim_{\epsilon \to 0^+} \int_{\{y:|y|>\delta/\epsilon\}} |\phi(y)|\, dy = 0.$$

Thus, we conclude that

$$\limsup_{\epsilon \to 0^+} |\phi_\epsilon * f(x) - f(x) \int \phi(y)\, dy| \leq \eta \int |\phi|\, dy.$$

Since $\eta > 0$ is arbitrary, the conclusion of the lemma follows. ∎

Theorem 5.14 *If ϕ has radial non-increasing majorant in L^1, and f is in L^p for some p, $1 \leq p \leq \infty$, then for a.e. $x \in \mathbf{R}^n$,*

$$\lim_{\epsilon \to 0^+} \phi_\epsilon * f(x) = f(x) \int \phi\, dx.$$

Proof. The proof for $p = 1$, $1 < p < \infty$ and $p = \infty$ are each slightly different.

Let $\theta(f)(x) = \limsup_{\epsilon \to 0^+} |\phi_\epsilon * f(x) - f(x) \int \phi|$. Our goal is to show that $\theta(f) = 0$ a.e. Observe that according to Lemma 5.13, we have if g is continuous and bounded, then

$$\theta(f) = \theta(f - g).$$

Also, according to Proposition 5.12, we have that there is a constant C so that with $I = |\int \phi|$,

$$\theta(f-g)(x) \leq |f(x) - g(x)|I + CM(f-g)(x). \tag{5.15}$$

If f is in L^1 and $\lambda > 0$, we have that for any bounded and continuous g that

$$\begin{aligned} m(\{x : \theta(f)(x) > \lambda\}) &\leq m(\{x : \theta(f-g)(x) > \lambda/2\}) + m(\{x : I|f(x) - g(x)| > \lambda/2\}) \\ &\leq \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x) - g(x)|\, dx. \end{aligned}$$

The first inequality uses (5.15) and the second uses the weak-type 1,1 property of the maximal function and Tchebishev. Since we can approximate f in the L^1 norm by functions g which are bounded and continuous, we conclude that $m(\{x : \theta(x) > \lambda\}) = 0$. Since this holds for each $\lambda > 0$, then we have that $m(\{x : \theta(x) > 0\}) = 0$

If f is in L^p, $1 < p < \infty$, then we can argue as above and use the that the maximal operator is of strong-type p, p to conclude that for any continuous and bounded g,

$$m(\{x : \theta(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int |f(x) - g(x)|^p\, dx.$$

Again, continuous and bounded functions are dense in L^p, if $p < \infty$ so we can conclude $\theta(f) = 0$ a.e.

Finally, if $p = \infty$, we claim that for each natural number, n, the set $\{x : \theta(f)(x) > 0 \text{ and } |x| < n\}$ has measure zero. This implies the theorem. To establish the claim, we write $f = \chi_{B_{2n}(0)} f + (1 - \chi_{B_{2n}(0)})f = f_1 + f_2$. Since f_1 is in L^p for each p finite, we have $\theta(f_1) = 0$ a.e. and it is easy to see that $\theta(f_2)(x) = 0$ if $|x| < 2n$. Since $\theta(f)(x) \leq \theta(f_1)(x) + \theta(f_2)(x)$, the claim follows. ∎

The standard Lebesgue differentiation theorem is a special case of the result proved above.

Corollary 5.16 *If f is in $L^1_{loc}(\mathbf{R}^n)$, then*

$$f(x) = \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\, dy.$$

Corollary 5.17 *If f is in $L^1_{loc}(\mathbf{R}^n)$, then there is a measurable set E, with $\mathbf{R}^n \setminus E$ of Lebesgue measure 0 and so that*

$$\lim_{r\to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0, \qquad x \in E.$$

We omit the proof of this last Corollary.

The set E from the previous theorem is called the Lebesgue set of f. It is clear from the definition that the choice of the representative of f may change E by a set of measure zero.

Chapter 6

Singular integrals

In this section, we will introduce a class of symbols for which the multiplier operators introduced in Chapter 3 are also bounded on L^p. The operators we consider are modeled on the Hilbert transform and the Riesz transforms. They were systematically studied by Calderón and Zygmund in the 1950's and are typically called Calderón-Zygmund operators. These operators are (almost) examples of pseudo-differential operators of order zero. The distinction between Calderón Zygmund operators and pseudo-differential operators is the viewpoint from which the operators are studied. If one studies the operator as a convolution operator, which seems to be needed to make estimates in L^p, then one is doing Calderón Zygmund theory. If one is studying the operator as a multiplier, which is more efficient for computing inverses and compositions, then one is studying pseudo-differential operators. One feature of pseudo-differential operators is that there is a general flexible theory for variable coefficient symbols. Our symbols will only depend on the frequency variable ξ.

6.1 Calderón-Zygmund kernels

In this chapter, we will consider linear operators $T : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$. In addition, we assume that the operator T has a kernel $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ which gives the action of T away from the diagonal. The kernel K is a function which is locally integrable on $\mathbf{R}^n \times \mathbf{R}^n \setminus \{(x,y) : x = y\}$. That K gives the action of T away from the diagonal means that that for any two functions f and g in $\mathcal{D}(\mathbf{R}^n)$ and which have disjoint support, we have that

$$Tf(g) = \int_{\mathbf{R}^{2n}} K(x,y)f(y)g(x)\,dx\,dy. \qquad (6.1)$$

Note that the left-hand side of this equation denotes the distribution Tf paired with the function g. We say that K is a *Calderón-Zygmund kernel* if there is a constant C_K so that K satisfies the following two estimates:

$$|K(x,y)| \leq \frac{C_K}{|x-y|^n} \tag{6.2}$$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{C_K}{|x-y|^{n+1}} \tag{6.3}$$

Exercise 6.4 *Show that the kernel is uniquely determined by the operator.*

Exercise 6.5 *What is the kernel of the identity operator?*

Exercise 6.6 *Let α be a multi-index. What is the kernel of the operator*

$$T\phi = \frac{\partial^\alpha \phi}{\partial x^\alpha}?$$

Conclude that the operator is not uniquely determined by the kernel.

If an operator T has a Calderón-Zygmund kernel K as described above and T is L^2 bounded, then T is said to be a *Calderon-Zygmund operator.* In this chapter, we will prove two main results. We will show that Calderón-Zygmund operators are also L^p-bounded, $1 < p < \infty$ and we will show that a large class of multipliers operators are Calderón-Zygmund operators.

Since Calderón-Zygmund kernels are locally bounded in the complement of $\{(x,y) : x = y\}$, if f and g are L^2 and have disjoint compact supports, then (6.1) continues to hold. To see this we approximate f and g by smooth functions and note that we can arrange that we only increase the support by a small amount when we approximate.

Exercise 6.7 *Suppose that Ω is a smooth function near the sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$, then show that*

$$K(x,y) = \Omega\left(\frac{x-y}{|x-y|}\right)\frac{1}{|x-y|^n}$$

is a Calderón-Zygmund kernel.

Exercise 6.8 *If $n \geq 3$ and j, k are in $\{1, \ldots, n\}$, then*

$$\frac{\partial^2}{\partial x_j \partial x_k}\frac{1}{|x-y|^{n-2}}$$

is a Calderón-Zygmund kernel. Of course, this result is also true for $n = 2$, but it is not very interesting.

In two dimensions, show that for any j and k,

$$\frac{\partial^2}{\partial x_j \partial x_k} \log|x - y|$$

is a Calderón-Zygmund kernel.

Exercise 6.9 *Let $z = z_1 + iz_2$ give a point in the complex plane. Show that the kernel*

$$K(z, w) = \frac{1}{(z - w)^2}$$

gives a Calderón-Zygmund kernel on $\mathbf{R}^2$.

Theorem 6.10 *If T is a Calderón-Zygmund operator, then for $1 < p < \infty$ there is a constant C so that*

$$\|Tf\|_p \leq C\|f\|_p.$$

The constant $C \leq A \max(p, p')$ where A depends on the dimension n, the constant in the estimates for the Calderón-Zygmund kernel, and the norm of T as an operator on L^2.

The main step of the proof is to prove a weak-type 1,1 estimate for T and then to interpolate to obtain the range $1 < p < 2$. The range $2 < p < \infty$ follows by applying the first case to the adjoint of T.

Exercise 6.11 *Let $\mathcal{H}$ be a Hilbert space with inner product $\langle , \rangle$ If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear map on a Hilbert space, then the map $x \to \langle Tx, y \rangle$ defines a linear functional on $\mathcal{H}$. Hence, there is a unique element y^* so that $\langle Tx, y \rangle = \langle x, y^* \rangle$.*

a) Show that the map $y \to y^* = T^*y$ is linear and bounded. The map T^* is called the adjoint *of the operator T.*

b) Suppose now that T is bounded on the Hilbert space L^2, and that, in addition to being bounded on L^2, the map T satisfies $\|Tf\|_p \leq A\|f\|_p$, say for all f in L^2. Show that $\|T^*f\|_{p'} \leq A\|f\|_{p'}$.

Exercise 6.12 *If $T : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space, show that we have $\|T\|_{\mathcal{L}(\mathcal{H})} = \|T^*\|_{\mathcal{L}(\mathcal{H})}$.*

Exercise 6.13 *If T is a Calderón-Zygmund operator with kernel K, show that T^* is also a Calderón-Zygmund operator and that the kernel of T^* is*

$$K^*(x,y) = \bar{K}(y,x).$$

Exercise 6.14 *If T_m is a multiplier operator with bounded symbol, show that the adjoint is a multiplier operator with symbol $\bar{m}$, $T_m^* = T_{\bar{m}}$.*

Our next theorem gives a weak-type 1,1 estimate for Calderón-Zygmund operators.

Theorem 6.15 *If T is a Calderón-Zygmund operator, f is in $L^2(\mathbf{R}^n)$ and $\lambda > 0$, then*

$$m(\{x : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} |f(x)|\,dx.$$

This result depends on the following decomposition lemma for functions. In this Lemma, we will use *cubes* on $\mathbf{R}^n$. By a cube, we mean a set of the form $Q_h(x) = \{y : |x_j - y_j| \leq h/2\}$. We let $\mathcal{D}_0$ be the mesh of cubes with side-length 1 and whose vertices have integer coordinates. For k an integer, we define $\mathcal{D}_k$ to be the cubes obtained by applying the dilation $x \to 2^k x$ to each cube in $\mathcal{D}_0$. The cubes in $\mathcal{D}_k$ have side-length 2^k and are obtained by bisecting each of the sides of the cubes in $\mathcal{D}_{k-1}$. Thus, if we take any two cubes Q and Q', in $\mathcal{D} = \cup_k \mathcal{D}_k$, then either one is contained in the other, or the two cubes have disjoint interiors. Also, given a cube Q, we will refer to the 2^n cubes obtained by dividing Q as the children of Q. And of course, if Q is a child of Q', then Q' is a parent of Q. The collection of cubes $\mathcal{D}$ will be called the *dyadic cubes* on $\mathbf{R}^n$.

Lemma 6.16 *(Calderón-Zygmund decomposition) If $f \in L^1(\mathbf{R}^n)$ and $\lambda > 0$, then we can find a family of cubes $\{Q_k\}_{k=1}^{\infty}$ with disjoint interiors so that $|f(x)| \leq \lambda$ a.e. in $\mathbf{R}^n \setminus \cup_k Q_k$ and for each cube we have*

$$\lambda < \frac{1}{m(Q_k)}\int_{Q_k} |f(x)|\,dx \leq 2^n\lambda.$$

As a consequence, we can write $f = g + b$ where $|g(x)| \leq 2^n\lambda$ a.e. and $b = \sum b_k$ where each b_k is supported in one of the cubes Q_k, each b_k has mean value zero $\int b_k = 0$ and satisfies $\|b_k\|_1 \leq 2\int_{Q_k} |f|\,dx$. The function g satisfies $\|g\|_1 \leq \|f\|_1$

Proof. Given $f \in L^1$ and $\lambda > 0$, we let $\mathcal{E}$ be the collection of cubes $Q \in \mathcal{D}$ which satisfy the inequality

$$\frac{1}{m(Q)}\int_Q |f(x)|\,dx > \lambda. \tag{6.17}$$

Note that because $f \in L^1$, if $m(Q)^{-1}\|f\|_1 \leq \lambda$, then the cube Q will not be in $\mathcal{E}$. That is $\mathcal{E}$ does not contain cubes of arbitrarily large side-length. Hence, for each cube Q' in $\mathcal{E}$, there is a largest cube Q in $\mathcal{E}$ which contains Q'. We let these maximal cubes form the collection $\mathcal{M} = \{Q_k\}$, which we index in an arbitrary way. If Q'_k is the parent of some Q_k in $\mathcal{M}$, then Q'_k is not in $\mathcal{E}$ and hence the inequality (6.17) fails for Q'_k. This implies that we have

$$\int_{Q_k} |f(x)|\,dx \leq \int_{Q'_k} |f(x)| \leq 2^n m(Q_k)\lambda. \tag{6.18}$$

Hence, the stated conditions on the family of cubes hold.

For each selected cube, Q_k, we define $b_k = (f - m(Q_k)^{-1}\int_{Q_k} f(x)\,dx)\chi_{Q_k}$ on Q_k and zero elsewhere. We set $b = \sum_k b_k$ and then $g = f - b$. It is clear that $\int b_k = 0$. By the triangle inequality,

$$\int |b_k(x)|\,dx \leq 2\int_{Q_k} |f(x)|\,dx.$$

It is clear that $\|g\|_1 \leq \|f\|_1$. We verify that $|g(x)| \leq 2^n\lambda$ a.e. On each cube Q_k, this follows from the upper bound for the average of $|f|$ on Q_k. For each x in the complement of $\cup_k Q_k$, there is sequence of cubes in $\mathcal{D}$, with arbitrarily small side-length and which contain x where the inequality (6.17) fails. Thus, the Lebesgue differentiation theorem implies that $|g(x)| \leq \lambda$ a.e. ∎

Our next step in the proof is the following Lemma regarding the kernel.

Lemma 6.19 *If K is a Calderón-Zygmund kernel and x, y are in $\mathbf{R}^n$ with $|x-y| \leq d$, then*

$$\int_{\mathbf{R}^n \setminus B_{2d}(x)} |K(z,x) - K(z,y)|\,dz \leq C.$$

The constant depends only on the dimension and the constant appearing in the definition of the Calderón-Zygmund kernel.

Proof. We apply the mean-value theorem of calculus to conclude that if $y \in \bar{B}_d(x)$ and $z \in \mathbf{R}^n \setminus B_{2d}(x)$, then the kernel estimate (6.3) implies

$$|K(z,x) - K(z,y)| \leq |x-y| \sup_{y\in B_d(x)} |\nabla_y K(z,y)| \leq 2^{n+1}C_K|x-y||x-z|^{-n-1}. \tag{6.20}$$

The second inequality uses the triangle inequality $|y-z| \geq |x-z| - |y-x|$ and then that $|x-z| - |y-x| \geq |x-z|/2$ if $|x-y| \leq d$ and $|x-z| \geq 2d$. Finally, if we integrate the inequality (6.20) in polar coordinates, we find that

$$\int_{\mathbf{R}^n\setminus B_{2d}(x)} |K(z,x) - K(z,y)|\,dz \leq dC_K 2^{n+1}\omega_{n-1}\int_{2d}^{\infty} r^{-n-1}r^{n-1}\,dr = C_K 2^n \omega_{n-1}.$$

This is the desired conclusion. ∎

Now, we give the proof of the weak-type 1,1 estimate in Theorem 6.15.

Proof of Theorem 6.15. We may assume that f is in $L^1 \cap L^2$. We let $\lambda > 0$. We apply the Calderón-Zygmund decomposition, Lemma 6.16 at λ to write $f = g + b$. We have

$$\{x : |Tf(x)| > \lambda\} \subset \{x : |Tg(x)| > \lambda/2\} \cup \{x : |Tb(x)| > \lambda/2\}.$$

Using Tchebisheff's inequality, the L^2-boundedness of T, and then that $|g(x)| \leq 2^n\lambda$ we obtain

$$m(\{x : |Tg(x)| > \lambda/2\}) \leq \frac{C}{\lambda^2}\int_{\mathbf{R}^n} |g(x)|^2\,dx \leq \frac{C}{\lambda}\int_{\mathbf{R}^n} |g(x)|\,dx$$

Finally, since $\|g\|_1 \leq \|f\|_1$, we have

$$m(\{x : |Tg(x)| > \lambda/2\}) \leq \frac{C}{\lambda}\|f\|_1.$$

Now, we turn to the estimate of Tb. We let $O_\lambda = \cup B_k$ where each ball B_k is chosen to have center x_k, the center of the cube Q_k and the radius of B_k is $\sqrt{n}$ multiplied by the side-length of Q. Thus, if $y \in Q_k$, then the distance $|x_k - y|$ is at most half the radius of B_k. This will be needed to apply Lemma 6.19. We estimate the measure of O_λ using that

$$m(O_\lambda) \leq C\sum_k m(Q_k) \leq \frac{1}{\lambda}\sum_k \int_{Q_k} |f|\,dx \leq \frac{C}{\lambda}\|f\|_1.$$

Next, we obtain an L^1 estimate for Tb_k. If x is in the complement of Q_k, we know that $Tb_k(x) = \int K(x,y)b_k(y)\,dy = \int (K(x,y) - K(x,x_k))b_k(y)\,dy$ where the second equality uses that b_k has mean value zero. Now, applying Fubini's theorem and Lemma 6.19, we can estimate

$$\begin{aligned}\int_{\mathbf{R}^n\setminus B_k} |Tb_k(x)|\,dx &\leq \int_{Q_k} |b_k(y)| \int_{\mathbf{R}^n\setminus B_k} |K(x,y) - K(x,x_k)|\,dx\,dy \\ &\leq C\int_{Q_k} |b_k(y)|\,dy \leq C\int_{Q_k} |f(y)|\,dy\end{aligned}$$

Thus, if we add on k, we obtain

$$\int_{\mathbf{R}^n\setminus O_\lambda} |Tb(y)|\,dy \leq \sum_k \int_{\mathbf{R}^n\setminus B_k} |Tb_k(y)|\,dy \leq C\|f\|_1 \qquad (6.21)$$

Finally, we estimate

$$\begin{aligned} m(\{x : Tb(x) > \lambda/2\}) &\leq m(O_\lambda) + m(\{x \in \mathbf{R}^n \setminus O_\lambda : |Tb(x)| > \lambda/2\}) \\ &\leq \frac{C}{\lambda}\|f\|_1. \end{aligned}$$

Where the the last inequality uses Chebishev and our estimate (6.21) for the L^1-norm of Tb in the complement of O_λ. ∎

Exercise 6.22 *Let Q be a cube in $\mathbf{R}^n$ of side-length $h > 0$, $Q = \{x : 0 \leq x_i \leq h\}$. Compute the diameter of Q. Hint: The answer is probably $h\sqrt{n}$.*

Proof of Theorem 6.10. Since we assume that T is L^2-bounded, the result for $1 < p < 2$, follows immediately from Theorem 6.15 and the Marcinkiewicz interpolation theorem, Theorem 4.19. The result for $2 < p < \infty$ follows by observing that if T is a Calderón-Zygmund operator, then the adjoint T^* is also a Calderón-Zygmund operator and hence T^* is L^p-bounded, $1 < p < 2$. Then it follows that T is L^p-bounded for $2 < p < \infty$.

The alert reader might observe that Theorem 4.19 appears to give a bound for the operator norm which grows like $|p-2|^{-1}$ near $p = 2$. This growth is a defect of the proof and is not really there. To see this, one can pick one's favorite pair of exponents, say 4/3 and 4 and interpolate (by either Riesz-Thorin or Marcinkiewicz) between them to see that norm is bounded for p near 2. ∎

Exercise 6.23 *Suppose that T is an operator on functions on the real line that is given off the diagonal by the kernel $K(x, y) = 1/(x - y)$. Show that we may not have Tf in L^1. Hint: Let $f(x)$ be smooth, non-negative function which is supported in $(-1, 1)$. What can you say about $Tf(x)$ for $x > 2$?*

6.2 Some multiplier operators

In this section, we study multiplier operators where the symbol m is smooth in the complement of the origin. For each $k \in \mathbf{R}$, we define a class of multipliers which we call symbols of order k. We say m is symbol of order k if for each multi-index α, there exists a constant C_α so that

$$\left|\frac{\partial^\alpha m}{\partial \xi^\alpha}(\xi)\right| \leq C_\alpha |\xi|^{-|\alpha|+k}. \tag{6.24}$$

The operator given by a symbol of order k corresponds to a generalization of a differential operator of order k. Strictly speaking, these operators are not pseudo-differential operators because we allow symbols which are singular near the origin. The symbols we study transform nicely under dilations. This makes some of the arguments below more elegant, however the inhomogeneous theory is probably more useful.

Exercise 6.25 *a) If $P(\xi)$ is homogeneous polynomial of degree k, then P is a symbol of order k.*

b) The multiplier for the Bessel potential operator $(1+|\xi|^2)^{-s/2}$ is a symbol of order $-s$ for $s \geq 0$. What if $s < 0$?

We begin with a lemma to state some of the basic properties of these symbols.

Lemma 6.26 *a) If m_j is a symbol of order k_j for $j = 1, 2$, then $m_1 m_2$ is a symbol of order $k_1 + k_2$ and each constant for $m_1 m_2$ depends on finitely many of the constants for m_1 and m_2.*

b) If $\eta \in \mathcal{S}(\mathbf{R}^n)$, then η is a symbol of order k for any $k \leq 0$.

c) If m is a symbol of order k, then for all $\epsilon > 0$, $\epsilon^{-k} m(\epsilon\xi)$ is a symbol of order k and the constants are independent of ϵ.

d) If m_j, $j = 1, 2$ are symbols of order k, then $m_1 + m_2$ is a symbol of order k.

Proof. A determined reader armed with the Leibniz rule will find that these results are either easy or false. ∎

Exercise 6.27 *a) Use Lemma 6.26 to show that if m is a symbol of order 0 and $\eta \in \mathcal{S}(\mathbf{R}^n)$ with $\eta = 1$ near the origin, then $m_\epsilon(\xi) = \eta(\epsilon\xi)(1 - \eta(\xi/\epsilon))m(\xi)$ is a symbol of order 0.*

b) Show that if $\eta(0) = 1$, then for each $f \in L^2(\mathbf{R}^n)$ the multiplier operators given by m and m_ϵ satisfy

$$\lim_{\epsilon \to 0^+} \|T_m f - T_{m_\epsilon} f\|_2 = 0.$$

c) Do we have $\lim_{\epsilon\to 0^+} \|T_m - T_{m_\epsilon}\| = 0$? Here, $\|T\|$ denotes the operator norm of T as an operator on L^2.

Exercise 6.28 *Show that if m is a symbol of order 0 and there is a $\delta > 0$ so that $|m(\xi)| \geq \delta$ for all $\xi \neq 0$, then m^{-1} is a symbol of order 0.*

Lemma 6.29 *If m is in the Schwartz class and m is a symbol of order $k > -n$, then there is a constant C depending only on finitely many of the constants in (6.24) so that*

$$|\check{m}(x)| \leq C|x|^{-n-k}.$$

Proof. To see this, introduce a cutoff function $\eta_0 \in \mathcal{D}(\mathbf{R}^n)$ which satisfies $\eta_0(\xi) = 1$ if $|\xi| < 1$ and $\eta_0(\xi) = 0$ if $|\xi| > 2$. Also, we set $\eta_\infty = 1 - \eta_0$. We write

$$K_j(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \eta_j(\xi|x|) m(\xi)\, d\xi, \quad j = 0, \infty.$$

For $j = 0$, the estimate is quite simple since $\eta_0(\xi|x|) = 0$ if $|\xi| > 2/|x|$. Thus,

$$|K_0(x)| \leq (2\pi)^{-n} \int_{|\xi|<2/|x|} |\xi|^k \, d\xi = C|x|^{-k-n}.$$

For the part near ∞, we need to take advantage of the cancellation that results from integrating the oscillatory exponential against the smooth symbol m. Thus, we write $(ix)^\alpha e^{ix\cdot\xi} = \frac{\partial^\alpha}{\partial\xi^\alpha} e^{ix\cdot\xi}$ and then integrate by parts to obtain

$$(2\pi)^n (ix)^\alpha K_\infty(x) = \int_{\mathbf{R}^n} (\frac{\partial^\alpha}{\partial\xi^\alpha} e^{ix\cdot\xi}) \eta_\infty(\xi|x|) m(\xi)\, d\xi = (-1)^{|\alpha|} \int e^{ix\cdot\xi} \frac{\partial^\alpha}{\partial\xi^\alpha} (\eta_\infty(\xi|x|) m(\xi))\, d\xi.$$

The boundary terms vanish since the integrand is in the Schwartz class. Using the symbol estimates (6.24) and that η_∞ is zero for $|\xi|$ near 0, we have for $k - |\alpha| < -n$, that

$$|(ix)^\alpha K_\infty(x)| \leq C \int_{|\xi|>1/|x|} |\xi|^{k-|\alpha|}\, d\xi = C|x|^{-n-k+|\alpha|}.$$

This implies the desired estimate that $|K_\infty(x)| \leq C|x|^{-n-k}$. ∎

We are now ready to show that the symbols of order 0 give Calderón Zygmund operators.

Theorem 6.30 *If m is a symbol of order 0, then T_m is a Calderón-Zygmund operator.*

Proof. The L^2-boundedness of T_m is clear since m is bounded, see Theorem 3.7. We will show that the kernel of T_m is of the form $K(x - y)$ and that for all multi-indices α there is a constant C_α so that K satisfies

$$|\frac{\partial^\alpha}{\partial x^\alpha} K(x)| \leq C|x|^{-n-|\alpha|}.$$

The inverse Fourier transform of m, $\check{m}$, is not, in general, a function. Thus, it is convenient to approximate m by nice symbols. To do this, we let $\eta \in \mathcal{D}(\mathbf{R}^n)$ satisfy $\eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ if $|x| > 2$. We define $m_\epsilon(\xi) = \eta(\epsilon\xi)(1 - \eta(\xi/\epsilon))m(\xi)$.

By Lemma 6.26, we see that m_ϵ is a symbol with constants independent of ϵ. Since $m_\epsilon \in \mathcal{S}(\mathbf{R}^n)$, by Lemma 6.29 we have that $K_\epsilon = \check{m}_\epsilon$ satisfies for each multi-index α,

$$|\frac{\partial^\alpha}{\partial x^\alpha} K_\epsilon(x)| \leq C|x|^{-n-|\alpha|}. \tag{6.31}$$

This is because the derivative of order α of K_ϵ by Proposition 1.19 is the inverse Fourier transform of $(-i\xi)^\alpha m_\epsilon(\xi)$, a symbol of order $|\alpha|$. Since the constants in the estimates are uniform in ϵ, we can apply the Arzela-Ascoli theorem to prove that there is a sequence $\{\epsilon_j\}$ with $\lim_{j\to\infty} \epsilon_j = 0$ so that K_{ϵ_j} and all of its derivatives converge uniformly on compact subsets of $\mathbf{R}^n \setminus \{0\}$ and of course the limit, which we call K, satisfies the estimates (6.31).

It remains to show that $K(x-y)$ is a kernel for the operator T_m. Let f be in $\mathcal{S}(\mathbf{R}^n)$. By the dominated convergence theorem and the Plancherel theorem, $T_{m_\epsilon} f \to T_m f$ in L^2 as $\epsilon \to 0^+$. By Proposition 1.24, $T_{m_\epsilon} f = K_\epsilon * f$. Finally, if f and g have disjoint support, then

$$\begin{aligned}
\int T_m f(x) g(x)\, dx &= \lim_{j\to\infty} \int T_{m_{\epsilon_j}} f(x) g(x)\, dx \\
&= \lim_{j\to\infty} \int K_{\epsilon_j}(x-y) f(y) g(x)\, dx\, dy \\
&= \int K(x-y) f(y) g(x)\, dx\, dy.
\end{aligned}$$

The first equality above holds because $T_{m_\epsilon} f$ converges in L^2, the second follows from Proposition 1.24 and the third equality holds because of the locally uniform convergence of K in the complement of the origin. This completes the proof that $K(x-y)$ is a kernel for T_m. ∎

We can now state a corollary which is usually known as the Mikhlin multiplier theorem.

Corollary 6.32 *If m is a symbol of order 0, then the multiplier operator T_m is bounded on L^p for $1 < p < \infty$.*

We conclude with a few exercises.

Exercise 6.33 *If m is infinitely differentiable in $\mathbf{R}^n \setminus \{0\}$ and is homogeneous of degree 0, then m is a symbol of order zero.*

Exercise 6.34 *We consider the symbol* $m(\xi) = -i\,\mathrm{sign}(\xi)$ *on the real line. Let* ϕ *be a smooth function which* 0 *if* $t < 1$ *and* 1 *if* $t > 2$ *and set* $m_\epsilon(\xi) = \phi(|\xi|/\epsilon)(1-\phi(\epsilon|\xi|))m(\xi)$. *Let* H_ϵ *be the multiplier operator given by* $H_\epsilon f = (m_\epsilon \hat{f})\check{}$ *and let* $k_\epsilon = \check{m}_\epsilon$.

a) Show that $k_\epsilon(x) \leq C/|x|$.

b) Show that k_ϵ *is an odd function.*

c) Show that

$$\lim_{\epsilon\to 0^+} k_\epsilon(y) = \frac{1}{\pi y}.$$

d) Show that for f *in* $\mathcal{S}(\mathbf{R})$ *and all* $x \in \mathbf{R}$, *we have*

$$\lim_{\epsilon\to 0^+} k_\epsilon * f(x) = \lim_{\epsilon\to 0^+} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\,dy.$$

Exercise 6.35 *Let* u *be the principal value distribution on the complex plane given by*

$$u(f) = \lim_{\epsilon\to 0^+} \int_{|z|>\epsilon} \frac{f(z)}{z^2}\,dz.$$

Find $\hat{u}$. *(I am not sure how hard this one will be, let me know if you have a good hint.)*

In the next exercise, we introduce the *Laplacian* $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.

Exercise 6.36 *Let* $1 < p < \infty$, $n \geq 3$. *If* $f \in \mathcal{S}(\mathbf{R}^n)$, *then we can find a tempered distribution* u *so that* $\Delta u = f$ *and we have the estimate*

$$\|\frac{\partial^2 u}{\partial x_j \partial x_k}\|_p \leq C\|f\|_p$$

where the constant in the estimate C *depends only on* p *and* n. *Why is* $n = 2$ *different? In two dimensions, show that we can construct* u *if* $\hat{f}(0) = 0$. *(This construction can be extended to all of the Schwartz class, but it is more delicate when* $\hat{f}(0) \neq 0$.*)*

This exercise gives an estimate for the solution of $\Delta u = f$. The estimate follows immediately from our work so far. We should also prove uniqueness: If u is a solution of $\Delta u = 0$ and u has certain growth properties, then $u = 0$. This is a version of the Liouville theorem. The above inequality is not true for every solution of $\Delta u = f$. For example, on $\mathbf{R}^2$, if $u(x) = e^{x_1 + ix_2}$, then we have $\Delta u = 0$, but the second derivatives are not in any $L^p(\mathbf{R}^2)$.

Exercise 6.37 *Let* $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ *be the wave operator which acts on functions of* $n+1$ *variables,* $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. *Can we find a solution of* $\Box u = f$ *and prove estimates like those in Exercise 6.36? Why or why not?*

Exercise 6.38 *Show that if* $\lambda \in \mathbf{C}$ *is not a negative real number, the operator given by* $m(\xi) = (\lambda + |\xi|^2)^{-1}$ *is bounded on* L^p *for* $1 < p < \infty$ *and that we have the estimate*

$$\|T_m f\|_p \leq C \|f\|_p.$$

Find the dependence of the constant on λ.

Using the machinery of singular integrals, we can extend our characterization of Sobolev spaces from Chapter 3 to $p \neq 2$.

Exercise 6.39 *For* $1 < p < \infty$ *and* $k = 0, 1, 2, \dots,$ *let* $L^{p,k}$ *be the Sobolev space of functions* f *so that for* $|\alpha| \leq k$, *the derivatives* $\partial^\alpha f / \partial x^\alpha$ *lie in* L^p.

For $1 < p < \infty$, *show that* $f \in L^{p,k}$ *if and only if* $((1+|\xi|^2)^{k/2}\hat{f})\check{}$ *lies in* L^p.

Chapter 7

Littlewood-Paley theory

In this chapter, we look at a particular singular integral and see how this can be used to characterize the L^p norm of a function in terms of its Fourier transform. The theory discussed here has its roots in the theory of harmonic functions in the disc or the upper half-plane. The expressions $Q_k f$ considered below, share many properties with the $2^{-k}\nabla u(x', 2^{-k})$ where u is the harmonic function in the upper-half plane $x_n > 0$ whose boundary values are f. Recently, many of these ideas have become part of the theory of wavelets. The operators $Q_k f$ decompose f into pieces which are of frequency approximately 2^k. A wavelet decomposition combines this decomposition in frequency with a spatial decomposition, in so far as this is possible.

7.1 A square function that characterizes L^p

We let ψ be a real-valued function in $\mathcal{D}(\mathbf{R}^n)$ which is supported in $\{\xi : 1/2 < |\xi| < 4\}$ and which satisfies $\sum_{k=-\infty}^{\infty} \psi_k(\xi)^2 = 1$ in $\mathbf{R}^n \setminus \{0\}$ where $\psi_k(\xi) = \psi(\xi/2^k)$ and we will call ψ a *Littlewood-Paley function.* It is not completely obvious that such a function exists.

Lemma 7.1 *A Littlewood-Paley function exists.*

Proof. We take a function $\tilde{\psi} \in \mathcal{D}(\mathbf{R}^n)$ which is non-negative, supported in $\{\xi : 1/2 < |\xi| < 4\}$ and which is strictly positive on $\{\xi : 1 < |\xi| < 2\}$. We set

$$\psi(\xi) = \tilde{\psi}(\xi)/(\sum_{k=-\infty}^{\infty} \tilde{\psi}^2(\xi/2^k))^{0.5}.$$

∎

For f in L^p, say, we can define $Q_k f = \check{\psi}_k * f = (\psi_k \hat{f})\check{}$. We define the *square function* $S(f)$ by

$$S(f)(x) = (\sum_{k=-\infty}^{\infty} |Q_k(f)(x)|^2)^{1/2}.$$

From the Plancherel theorem, Theorem 3.2, it is easy to see that

$$\|f\|_2 = \|S(f)\|_2 \tag{7.2}$$

and of course this depends on the identity $\sum_k \psi_k^2 = 1$. We are interested in this operator because we can characterize the L^p spaces in a similar way.

Theorem 7.3 *Let $1 < p < \infty$. There is a finite nonzero constant $C = C(p, n, \psi)$ so that if f is in L^p, then*

$$C_p^{-1}\|f\|_p \leq \|S(f)\|_p \leq C_p\|f\|_p.$$

This theorem will be proven by considering a vector-valued singular integral. The kernel we consider will be

$$K(x, y) = (\ldots, 2^{nk}\check{\psi}(2^k(x - y)), \ldots).$$

Lemma 7.4 *If ψ is in $\mathcal{S}(\mathbf{R}^n)$, then the kernel K defined above is a Calderón-Zygmund kernel.*

Proof. We write out the norm of K

$$|K(x, y)|^2 = \sum_{k=-\infty}^{\infty} 2^{2nk}|\check{\psi}((2^k(x - y))|^2.$$

We choose N so that $2^N \leq |x - y| < 2^{N+1}$ and split the sum above at $-N$. Recall that $\check{\psi}$ is in $\mathcal{S}(\mathbf{R}^n)$ and decays faster than any polynomial. Near 0, that is for $k \leq -N$, we use that $\check{\psi}(x) \leq C$. For $k > -N$, we use that $\check{\psi}(x) \leq C|x|^{-n-1}$. Thus, we have

$$|K(x, y)|^2 \leq C(\sum_{k=-\infty}^{-N} 2^{2nk} + \sum_{k=-N+1}^{\infty} 2^{2nk}(2^{k+N})^{-2(n+1)}) = C2^{-2nN}.$$

Recalling that 2^N is approximately $|x - y|$, we obtain the desired upper-bound for $K(x, y)$. To estimate the gradient, we observe that $\nabla_x K(x, y) = (\ldots, 2^{-(n+1)k}(\nabla\check{\psi})((x - y)2^k), \ldots)$. This time, we will need a higher power of $|x|$ to make the sum converge. Thus, we use that $|\nabla\check{\psi}(x)| \leq C$ near the origin and $|\nabla\check{\psi}(x)| \leq C|x|^{-n-2}$. This gives that

$$|\nabla K(x)|^2 \leq C(\sum_{k=-\infty}^{-N} 2^{2k(n+1)} + \sum_{k=-N+1}^{\infty} 2^{2k(n+1)}(2^{k+N})^{-2(n+2)}) = C2^{-2N(n+1)}.$$

Recalling that 2^N is approximately $|x - y|$ finishes the proof. ∎

Proof of Theorem 7.3. To establish the right-hand inequality, we fix N and consider the map $f \to (\psi_{-N}\hat{f}, \dots, \psi_N \hat{f})\check{} = K_N * f$. The kernel K_N is a vector-valued function taking values in the vector space $\mathbf{C}^{2N+1}$. We observe that the conclusion of Lemma 6.19 continues to hold, if we interpret the absolute values as the norm in the Hilbert space $\mathbf{C}^{2N+1}$, with the standard norm, $|(z_{-N}, \dots, z_N)| = (\sum_{k=-N}^{N} |z_k|^2)^{1/2}$.

As a consequence, we conclude that $K_N * f$ satisfies the L^p estimate of Theorem 6.10 and we have the inequality

$$\|(\sum_{k=-N}^{N} |Q_k f|^2)^{1/2}\|_p \leq \|f\|_p. \tag{7.5}$$

We can use the monotone convergence theorem to let $N \to \infty$ and obtain the right-hand inequality in the Theorem.

To obtain the other inequality, we argue by duality. First, using the polarization identity, we can show that for f, g in L^2,

$$\int_{\mathbf{R}^n} \sum_{k=-\infty}^{\infty} Q_k(f)(x)\overline{Q_k(g)}(x)\, dx = \int_{\mathbf{R}^n} f(x)\bar{g}(x)\, dx. \tag{7.6}$$

Next, we suppose that f is $L^2 \cap L^p$ and use duality to find the L^p norm of f, the identity (7.6), and then Cauchy-Schwarz and Hölder to obtain

$$\|f\|_p = \sup_{\|g\|_{p'}=1} \int_{\mathbf{R}^n} f(x)\bar{g}(x)\, dx = \sup_{\|g\|_{p'}=1} \int_{\mathbf{R}^n} \sum Q_k(f)(x)\overline{Q_k(g)}(x)\, dx \leq \|S(f)\|_p \|S(g)\|_{p'}.$$

Now, if we use the right-hand inequality, (7.5) which we have already proven, we obtain the desired conclusion. Note that we should assume g is in $L^2(\mathbf{R}^n) \cap L^{p'}(\mathbf{R}^n)$ to make use of the identity (7.2).

A straightforward limiting argument helps to remove the restriction that f is in L^2 and obtain the inequality for all f in L^p. ∎

7.2 Variations

In this section, we observe two simple extensions of the result above. These modifications will be needed in a later chapter.

For our next proposition, we consider an operators Q_k which are defined as above, except, that we work only in one variable. Thus, we have a function $\psi \in \mathcal{D}(\mathbf{R})$ and suppose that

$$\sum_{k=-\infty}^{\infty} |\psi(\xi_n/2^k)|^2 = 1.$$

We define the operator $f \to Q_k f = (\psi(\xi_n/2^k)\hat{f}(\xi))\check{}$.

Proposition 7.7 *If $f \in L^p(\mathbf{R}^n)$, then for $1 < p < \infty$, we have*

$$C_p \|f\|_p^p \leq \|(\sum_k |Q_k f|^2)^{1/2}\|_p^p \leq C_p \|f\|_p^p.$$

Proof. If we fix $x' = (x_1, \ldots, x_{n-1})$, then we have that

$$C_p \|f(x', \cdot)\|_{L^p(\mathbf{R})}^p \leq \|(\sum_k |Q_k f(x', \cdot)|^2)^{1/2}\|_p^p \leq C_p \|f(x', \cdot)\|_{L^p(\mathbf{R})}^p.$$

This is the one-dimensional version of Theorem 7.3. If we integrate in the remaining variables, then we obtain the Proposition. ∎

We will need the following Corollary for the one-dimensional operators. Of course the same result holds, with the same proof, for the n-dimensional operator.

Corollary 7.8 *If $2 \leq p < \infty$, then we have*

$$\|f\|_p \leq C(\sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2)^{1/2}.$$

If $1 < p \leq 2$, then we have

$$(\sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2)^{1/2} \leq C\|f\|_p.$$

Proof. To prove the first statement, we apply Minkowski's inequality bring the sum out through an $L^{p/2}$ norm to obtain

$$(\int_{\mathbf{R}^n} (\sum_{k=-\infty}^{\infty} |Q_k f(x)|^2)^{p/2}\, dx)^{2/p} \leq \sum_{k=-\infty}^{\infty} \|Q_k f\|_p^2.$$

The application of Minkowski's inequality requires that $p/2 \geq 1$. If we take the square root of this inequality and use Proposition 7.7, we obtain the first result of the Corollary.

The second result has a similar proof. To see this, we use Minkowski's integral inequality to bring the integral out through the $\ell^{2/p}$ norm to obtain

$$(\sum_{k=-\infty}^{\infty}(\int_{\mathbf{R}^n}|Q_k f(x)|^2\,dx)^{2/p})^{p/2} \le \int_{\mathbf{R}^n}(\sum_{k=-\infty}^{\infty}|Q_k f(x)|^2)^{p/2}.$$

Now, we may take the pth root and apply Proposition 7.7 to obtain the second part of our Corollary. ∎

Chapter 8

Fractional integration

In this chapter, we study the fractional integration operator or Riesz potential. To motivate these operators, we consider the following peculiar formulation of the fundamental theorem of calculus: If f is a nice function, then

$$f(x) = \int_{-\infty}^{x} f'(t)(x-t)^{1-1}\,dt.$$

Thus the map $g \to \int_{-\infty}^{x} g(t)\,dt$ is a left-inverse to differentiation. For $\alpha > 0$, we define a family of fractional integral operators by

$$I_\alpha^+ f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\,dt.$$

Exercise 8.1 *Show that if $\alpha > 0$and $\beta > 0$, then*

$$I_\alpha^+(I_\beta^+(f)) = I_{\alpha+\beta}^+(f).$$

In this section, we consider a family of similar operators in all dimensions. We will establish the L^p mapping properties of these operators. We also will consider the Fourier transform of the distribution given by the function $|x|^{\alpha-n}$. Using these results, we will obtain the Sobolev inequalities.

We begin by giving an example where these operators arise in nature. This exercise will be much easier to solve if we use the results proved below.

Exercise 8.2 *If f is in $\mathcal{S}(\mathbf{R}^n)$, then*

$$f(x) = \frac{1}{(2-n)\omega_{n-1}} \int_{\mathbf{R}^n} \Delta f(y)|x-y|^{2-n}\,dy.$$

Exercise 8.3 *For a point $x \in \mathbf{R}^2$, we let $x = x_1 + ix_2$ denote a point in $\mathbf{R}^2$. If f is in $\mathcal{D}(\mathbf{R}^2)$ and $\bar\partial f(x) = \frac{1}{2}(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2})$, show that*

$$f(x) = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{\bar\partial f(y)}{x-y}dy.$$

Hint: Verify that $\bar\partial 1/x = 0$ if $x \neq 0$ and then apply the divergence theorem in $\mathbf{R}^2 \setminus B_\epsilon(x)$. Let $\epsilon \to 0^+$.

8.1 The Hardy-Littlewood-Sobolev theorem

The operators we consider in $\mathbf{R}^n$ are the family of *Riesz potentials*

$$I_\alpha(f)(x) = \gamma(\alpha, n)\int_{\mathbf{R}^n} f(y)|x-y|^{\alpha-n} \tag{8.4}$$

for α satisfying $0 < \alpha < n$. The constant, $\gamma(\alpha, n)$ is given by

$$\gamma(\alpha, n) = \frac{2^{n-\alpha}\Gamma((n-\alpha)/2)}{(4\pi)^{n/2}\Gamma(\alpha/2)}.$$

The condition $\alpha > 0$ is needed in order to guarantee that $|x|^{\alpha-n}$ is locally integrable. Our main goal is to prove the L^p mapping properties of the operator I_α. We first observe that the homogeneity properties of this operator imply that the operator can map L^p to L^q only if $1/p - 1/q = \alpha/n$. By homogeneity properties, we mean: If $r > 0$ and we let $\delta_r f(x) = f(rx)$ be the action of dilations on functions, then we have

$$I_\alpha(\delta_r f) = r^{-\alpha}\delta_r(I_\alpha f). \tag{8.5}$$

This is easily proven by changing variables. This observation is essential in the proof of the following Proposition.

Proposition 8.6 *If the inequality*

$$\|I_\alpha f\|_q \leq C\|f\|_p$$

holds for all f in $\mathcal{S}(\mathbf{R}^n)$ and a finite constant C, then

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.$$

Proof. Observe that we have $\|\delta_r f\|_p = r^{-n/p}\|f\|_p$. This is proven by a change of variables if $0 < p < \infty$ and is obvious if $p = \infty$. (Though we will never refer to the case $p < 1$, there is no reason to restrict ourselves to $p \geq 1$.) Next, if f is in $\mathcal{S}(\mathbf{R}^n)$, then by (8.5)

$$\|I_\alpha(\delta_r f)\|_q = r^{-\alpha}\|\delta_r(I_\alpha f)\|_q = r^{-\alpha-n/q}\|I_\alpha f\|_q.$$

Thus if the hypothesis of our proposition holds, we have that for all Schwartz functions f and all $r > 0$, that

$$r^{-\alpha-n/q}\|I_\alpha f\|_q \leq C\|f\|_p r^{-n/p}.$$

If $\|I_\alpha f\|_q \neq 0$ then the truth of the above inequality for all $r > 0$ implies that the exponents on each side of the inequality must be equal. If $f \neq 0$ is non-negative, then $I_\alpha f > 0$ everywhere and hence $\|I_\alpha f\|_q > 0$ and we can conclude the desired relation on the exponents. ∎

Next, we observe that the inequality must fail at the endpoint $p = 1$. This follows by choosing a nice function with $\int \phi = 1$. Then with $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$, we have that as $\epsilon \to 0^+$,

$$I_\alpha(\phi_\epsilon)(x) \to \gamma(\alpha, n)|x|^{\alpha-n}.$$

If the inequality $\|I_\alpha \phi_\epsilon\|_{n/(n-\alpha)} \leq C\|\phi_\epsilon\|_1 = C$ holds uniformly as ϵ, then Fatou's Lemma will imply that $|x|^{\alpha-n}$ lies in $L^{n/(n-\alpha)}$, which is false.

Exercise 8.7 *Show that $I_\alpha : L^p \to L^q$ if and only if $I_\alpha : L^{q'} \to L^{p'}$. Hence, we can conclude that I_α does not map $L^{n/\alpha}$ to L^∞.*

Exercise 8.8 *Can you use dilations, δ_r to show that the inequality*

$$\|f * g\|_r \leq \|f\|_p\|g\|_q$$

can hold only if $1/r = 1/p + 1/q - 1$?

Exercise 8.9 *Show that the estimate*

$$\|\nabla f\|_p \leq C\|f\|_q$$

can not hold. That is if we fix p and q, there is no constant C so that the above inequality is true for all f in the Schwartz class. Hint: Let $f(x) = \eta(x)e^{i\lambda x_1}$ where η is a smooth bump.

We now give the positive result. The proof we give is due Lars Hedberg [16]. The result was first considered in one dimension (on the circle) by Hardy and Littlewood. The n-dimensional result was considered by Sobolev.

Theorem 8.10 *(Hardy-Littlewood-Sobolev) If $1/p - 1/q = \alpha/n$ and $1 < p < n/\alpha$, then there exists a constant $C = C(n, \alpha, p)$ so that*

$$\|I_\alpha f\|_q \leq C\|f\|_p.$$

The constant C satisfies $C \leq C(\alpha, n)\max((p-1)^{-(1-\frac{\alpha}{n})}, (\frac{1}{p} - \frac{\alpha}{n})^{-(1-\frac{\alpha}{n})})$.

Proof of Hardy-Littlewood-Sobolev inequality. We may assume that the L^p norm of f satisfies $\|f\|_p = 1$. We consider the integral defining I_α and break the integral into sets where $|x-y| < R$ and $|x-y| > R$:

$$I_\alpha f(x) \leq \gamma(\alpha, n)\left(\int_{B_R(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy + \int_{\mathbf{R}^n \setminus B_R(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy\right) \equiv \gamma(\alpha, n)(I + II).$$

By Proposition 5.12, we can estimate

$$I(x, R) \leq Mf(x)\omega_{n-1}\int_0^R r^{\alpha-n}r^{n-1}\,dr = Mf(x)\frac{R^\alpha}{\alpha}\omega_{n-1}$$

where we need that $\alpha > 0$ for the integral to converge. To estimate the second integral, $II(x, R)$, we use Hölder's inequality to obtain

$$\begin{aligned} II(x,R) \leq \|f\|_p\omega_{n-1}^{1/p'}\left(\int_{r>R} r^{(\alpha-n)p'+n-1}\,dr\right)^{1/p'} &= \|f\|_p\omega_{n-1}^{1/p'}\left(\frac{R^{(\alpha-n)p'+n}}{(n-\alpha)p'-n)}\right)^{1/p'} \\ &= \|f\|_p\omega_{n-1}^{1/p'}\frac{R^{\alpha-\frac{n}{p}}}{((n-\alpha)p'-n)^{1/p'}} \end{aligned}$$

where we need $\alpha < n/p$ for the integral in r to converge. Using the previous two inequalities and recalling that we have set $\|f\|_p = 1$, we have constants C_1 and C_2 so that

$$|I_\alpha(f)(x)| \leq C_1R^\alpha Mf(x) + C_2R^{\alpha-\frac{n}{p}}. \tag{8.11}$$

If we were dedicated analysts, we could obtain the best possible inequality (that this method will give) by differentiating with respect to R and using one-variable calculus to find the minimum value of the right-hand side of (8.11). However, we can obtain an answer which is good enough by choosing $R = Mf(x)^{-p/n}$. We substitute this value of R into (8.11) and obtain

$$|I_\alpha f(x)| \leq (C_1 + C_2) Mf(x)^{1-\alpha n/p}$$

and if we raise this inequality to the power $pn/(n - \alpha p)$ we obtain

$$|I_\alpha f(x)|^{np/(n-\alpha p)} \leq (C_1 + C_2)^{np/(n-\alpha p)} Mf(x)^p.$$

Now, if we integrate and use the Hardy-Littlewood theorem Theorem 5.11 we obtain the conclusion of this theorem. ∎

Exercise 8.12 *The dependence of the constants on p, α and n is probably not clear from the proof above. Convince yourself that the statement of the above theorem is correct.*

Exercise 8.13 *In this exercise, we use the behavior of the constants in the Hardy-Littlewood-Sobolev theorem, Theorem 8.10, to obtain an estimate at the endpoint, $p = n/\alpha$.*

Suppose that f is in $L^{n/\alpha}$ and $f = 0$ outside $B_1(0)$. We know that, in general, $I_\alpha f$ is not in L^∞. The following is a substitute result. Consider the integral

$$\int_{B_1(0)} \exp([\epsilon |I_\alpha f(x)|^{n/(n-\alpha)}])\, dx = \sum_{k=1}^{\infty} \frac{1}{k!} \epsilon^{nk/(n-k\alpha)} \int_{B_1(0)} |I_\alpha f(x)|^{\frac{kn}{n-\alpha}}\, dx.$$

Since f is in $L^{n/\alpha}$ and f is zero outside a ball of radius 1, we have that f is in $L^p(\mathbf{R}^n)$ for all $p < n/\alpha$. Thus, $I_\alpha f$ is in every L^q-space for $\infty > q > n/(n-\alpha)$. Hence, each term on the right-hand side is finite. Show that in fact, we can sum the series for ϵ small.

Exercise 8.14 *If α is real and $0 < \alpha < n$, show by example that I_α does not map $L^{n/\alpha}$ to L^∞. Hint: Consider a function f with $f(x) = |x|^{-\alpha}(-\log|x|)^{-1}$ if $|x| < 1/2$.*

Next, we compute the Fourier transform of the tempered distribution $\gamma(\alpha, n)|x|^{\alpha-n}$. More precisely, we are considering the Fourier transform of the tempered distribution

$$f \to \gamma(\alpha, n) \int_{\mathbf{R}^n} |x|^{\alpha-n} f(x)\, dx.$$

Theorem 8.15 *If $0 < \operatorname{Re}\alpha < n$, then*

$$\gamma(\alpha, n)(|x|^{\alpha-n})\hat{} = |\xi|^{-\alpha}.$$

Proof. We let $\eta(|\xi|)$ be a standard cutoff function which is 1 for $|\xi| < 1$ and 0 for $|\xi| > 2$. We set $m_\epsilon(\xi) = \eta(|\xi|\epsilon)(1-\eta(|\xi|/\epsilon))|\xi|^{-\alpha}$. The multiplier m_ϵ is a symbol of order $-\alpha$ uniformly in ϵ. Hence, by the result Lemma 6.29 of Chapter 6, we have that $K_\epsilon = \check{m}_\epsilon$ satisfies the estimates

$$|\frac{\partial^\beta}{\partial x^\beta} K_\epsilon(x)| \le C(\alpha,\beta)|x|^{\alpha-n-|\beta|}. \tag{8.16}$$

Hence, applying the Arzela-Ascoli theorem we can extract a sequence $\{\epsilon_i\}$ with $\epsilon_i \to 0$ so that K_{ϵ_i} converges uniformly to some function K on each compact subset of $\mathbf{R}^n \setminus \{0\}$. We choose f in $\mathcal{S}(\mathbf{R}^n)$ and recall the definition of the Fourier transform of a distribution to obtain

$$\begin{aligned} \int_{\mathbf{R}^n} K(x)\hat{f}(x)\,dx &= \lim_{j\to\infty} \int K_{\epsilon_j}(x)\hat{f}(x)\,dx \\ &= \lim_{j\to\infty} \int m_{\epsilon_j}(\xi) f(\xi)\,d\xi \\ &= \int |\xi|^{-\alpha} f(\xi)\,d\xi. \end{aligned}$$

The first equality depends on the uniform estimate for K_ϵ in (8.16) and the locally uniform convergence of the sequence K_{ϵ_j}. Thus, we have that $\hat{K}(\xi) = |\xi|^{-\alpha}$ in the sense of distributions. Note that each m_ϵ is radial. Hence, K_ϵ and thus K is radial. See Chapter 1.

Our next step is to show that the kernel K is homogeneous:

$$K(Rx) = R^{\alpha-n}K(x). \tag{8.17}$$

To see this, observe that writing $K = \lim_{j\to\infty} K_{\epsilon_j}$ again gives that

$$\begin{aligned} \int_{\mathbf{R}^n} K(Rx)\hat{f}(x)\,dx &= \lim_{j\to\infty} \int_{\mathbf{R}^n} K_{\epsilon_j}(Rx)\hat{f}(x)\,dx \\ &= R^{-n} \lim_{j\to\infty} \int_{\mathbf{R}^n} m_{\epsilon_j}(\xi/R) f(\xi)\,d\xi = R^{\alpha-n} \int_{\mathbf{R}^n} |\xi|^{\alpha-n} f(\xi)\,d\xi \\ &= R^{\alpha-n} \int K(x)\hat{f}(x)\,dx. \end{aligned}$$

This equality for all f in $\mathcal{S}(\mathbf{R}^n)$ implies that (8.17) holds. If we combine the homogeneity with the rotational invariance of K observed above, we can conclude that

$$\check{m}(x) = c|x|^{\alpha-n}.$$

It remains to compute the value of c. To do this, we only need to find one function where we can compute the integrals explicitly. We use the friendly gaussian. We consider

$$c\int |x|^{\alpha-n}e^{-|x|^2}\,dx = (4\pi)^{-n/2}\int |\xi|^{-\alpha}e^{-|\xi|^2/4}\,d\xi = 2^{n-\alpha}(4\pi)^{-n/2}\int |\xi|^{-\alpha}e^{-|\xi|^2}\,d\xi. \quad (8.18)$$

Writing the integrals in polar coordinates, substituting $s = r^2$, and then recalling the definition of the Gamma function, we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} |x|^{-\beta}e^{-|x|^2}\,dx &= \omega_{n-1}\int_0^\infty r^{n-\beta}e^{-r^2}\,\frac{dr}{r}\\ &= \frac{\omega_{n-1}}{2}\int_0^\infty s^{(n-\beta)/2}e^{-s}\,\frac{ds}{s}\\ &= \frac{1}{2}\Gamma(\frac{n-\beta}{2})\omega_{n-1}.\end{aligned}$$

Using this to evaluate the two integrals in (8.18) and solving for c gives

$$c = \frac{2^{n-\alpha}\Gamma((n-\alpha)/2)}{(4\pi)^{n/2}\Gamma(\alpha/2)}.$$

∎

We give a simple consequence.

Corollary 8.19 *For f in $\mathcal{S}(\mathbf{R}^n)$, we have*

$$I_\alpha(f) = (\hat{f}(\xi)|\xi|^{-\alpha})\check{}.$$

A reader who is not paying much attention, might be tricked into thinking that this is just an application of Proposition 1.24. Though I like to advocate such sloppiness, it is traditional to be a bit more careful. Note that Proposition 1.24 does not apply to the study of $I_\alpha f$ because $I_\alpha f$ is not the convolution of two L^1 functions. A proof could be given based on approximating the multiplier $|\xi|^{-\alpha}$ by nice functions. This result could have appeared in Chapter 2. However, we prefer to wait until it is needed to give the proof.

Proposition 8.20 *If u is a tempered distribution and f is a Schwartz function, then*

$$(f * u)\hat{} = \hat{f}\hat{u}.$$

Proof. Recall the definition for convolutions involving distributions that appeared in Chapter 2. By this and the definition of the Fourier transform and inverse Fourier transform, we have

$$(f * u)\check{}(g) = f * u(\hat{g}) = \check{\hat{u}}(\tilde{f} * \hat{g}) = \hat{u}((\tilde{f} * \hat{g})\check{}).$$

Now, we argue as in the proof of Proposition 1.24 and use the Fourier inversion theorem, 1.32 to obtain

$$(\tilde{f} * \hat{g})\check{}(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} f(\xi - \eta)\hat{g}(\eta) e^{ix\cdot((\xi-\eta+\eta))}\, d\xi d\eta = \hat{f}(x)g(x).$$

Thus, we have $(f * u)\check{}(g) = \hat{u}(\hat{f}g) = (\hat{f}\hat{u})(g)$. ∎

Proof of Corollary 8.19. This is immediate from Theorem 8.15 which gives the Fourier transform of the distribution given by $\gamma(\alpha, n)|x|^{\alpha-n}$ and the previous proposition. ∎

8.2 A Sobolev inequality

Next step is to establish an inequality relating the L^q-norm of a function f with the L^p-norm of its derivatives. This result, known as a *Sobolev inequality* is immediate from the Hardy-Littlewood-Sobolev inequality, once we have a representation of f in terms of its gradient.

Lemma 8.21 *If f is a Schwartz function, then*

$$f(x) = \frac{1}{\omega_{n-1}} \int_{\mathbf{R}^n} \frac{\nabla f(y) \cdot (x - y)}{|x - y|^n}\, dy.$$

Proof. We let $z' \in \mathbf{S}^{n-1}$ and then write

$$f(x) = -\int_0^\infty \frac{d}{dt} f(x + tz')\, dt = -\int_0^\infty z' \cdot (\nabla f)(x + tz')\, dt.$$

If we integrate both sides with respect to the variable z', and then change from the polar coordinates t and z' to y which is related to t and z' by $y - x = tz'$, we obtain

$$\omega_{n-1} f(x) = -\int_{\mathbf{S}^{n-1}} \int_0^\infty z' \cdot \nabla f(x+tz') t^{n-1+1-n}\, dt\, dz' = \int_{\mathbf{R}^n} \frac{x - y}{|x - y|} \cdot \nabla f(y) \frac{1}{|x - y|^{n-1}}\, dy.$$

This gives the conclusion. ∎

Theorem 8.22 *If $1 < p < n$ (and thus $n \geq 2$), f is in the Sobolev space $L^{p,1}$ and q is defined by $1/q = 1/p - 1/n$, then there is a constant $C = C(p,n)$ so that*

$$\|f\|_q \leq C\|\nabla f\|_p.$$

Proof. According to Lemma 8.21, we have that for nice functions,[1]

$$|f(x)| \leq I_1(|\nabla f|)(x).$$

Thus, the inequality of this theorem follows from the Hardy-Littlewood-Sobolev theorem on fractional integration. Since the Schwartz class is dense in the Sobolev space, a routine limiting argument extends the inequality to functions in the Sobolev space. ∎

The Sobolev inequality continues to hold when $p = 1$. However, the above argument fails. The standard argument for $p = 1$ is an ingenious argument due to Gagliardo, see Stein [29, pp. 128-130] or the original paper of Gagliardo [13].

Exercise 8.23 *If $p > n$, then the Riesz potential, I_1 produces functions which are in Hölder continuous of order $\gamma = 1 - (n/p)$. If $0 < \gamma < 1$, define the Hölder semi-norm by*

$$\|f\|_{C^\gamma} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma},$$

a) Show that if f is a Schwartz function, then $\|I_1(f)\|_{C^\gamma} \leq C\|f\|_p$ provided $p > n$ and $\gamma = 1 - (n/p)$. b) Generalize to I_α. c) The integral defining $I_1(f)$ is not absolutely convergent for all f in L^p if $p > n$. Show that the differences $I_1 f(x) - I_1 f(y)$ can be expressed as an absolutely convergent integral. Conclude that if $f \in L^{p,1}$, then $f \in C^\gamma$ for γ and p as above.

Exercise 8.24 *Show by example that the Sobolev inequality, $\|f\|_\infty \leq C\|\nabla f\|_n$ fails if $p = n$ and $n \geq 2$. Hint: For appropriate a, try f with $f(x) = \eta(x)(-\log|x|)^a$ with η a smooth function which is supported in $|x| < 1/2$.*

Exercise 8.25 *Show that there is a constant $C = C(n)$ so that if $g = I_1(f)$, then*

$$\sup_{r>0, x\in\mathbf{R}^n} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g(x) - (g)_{r,x}|\, dx \leq C\|f\|_n.$$

Here, $(f)_{r,x}$ denotes the average of f on the ball $B_r(x)$.

$$(f)_{r,x} = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\, dy.$$

[1]This assumes that $\omega_{n-1}^{-1} = \gamma(1,n)$, which I have not checked.

Exercise 8.26 *Show that in one dimension, we have the inequality* $\|f\|_\infty \le \|f'\|_1$ *for nice* f*. State precise hypotheses that* f *must satisfy.*

We now consider an operator in two-dimensions that will be of interest in a few chapters. In the next two exercises, we will use a complex variable $x = x_1 + ix_2$ to denote a point in $\mathbf{R}^2$.

Exercise 8.27 *We define the Cauchy transform of a Schwartz function by*

$$gf(x) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{f(y)}{x-y}\,dy.$$

Here, we are using dy *to denote the two-dimensional Lebesgue measure on the plane. a) Let* u *denote the distribution given by*

$$u(f) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{f(y)}{y}\,dy.$$

Verify that if we use the definition of the convolution of a tempered distribution and test function as in Chapter 3, we have

$$u * f(g) = \int g(f)(x)h(x)\,dx.$$

b) For f *in* $\mathcal{S}(\mathbf{C})$*, show that*

$$\bar{\partial} g(f)(x) = f(x).$$

c) If f *is in* L^p *for* $1 < p < 2$*, show that we have*

$$\bar{\partial} gf = f$$

where we identify gf *with the distribution* $h \to \int gf(x)h(x)\,dx$*.*

d) If f *is a Schwartz function, show that we have*

$$\partial g(f)(x) = \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_{|x-y|>\epsilon} \frac{f(y)}{(x-y)^2}\,dy.$$

Here, $\partial = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$*. We define the Beurling operator by* $Bf(x) = \partial g(f)(x)$*.*

e) Show that we have

$$\widehat{Bf}(\xi) = \frac{\bar{\xi}}{\xi}\hat{f}(\xi).$$

Again, we are using the complex variable $\xi = \xi_1 + i\xi_2$ *to denote a point in* $\mathbf{C}$. *Conclude that* $\|Bf\|_p \leq C_p\|f\|_p$, *for* $1 < p < \infty$.

f) Suppose f *is in* $\mathcal{S}(\mathbf{C})$. *Show that*

$$\int_{\mathbf{C}} |\bar{\partial} f|^2\, dx = \int_{\mathbf{C}} |\partial f|^2\, dx.$$

Hint: Integrate by parts.

Exercise 8.28 *The functions of bounded mean oscillation are the class of functions for which the expression*

$$\|f\|_* = \sup\{\int_B |f(x) - f_B|\, dx\}$$

is finite. Here, f_B *denotes the average of* f *on* B. *The supremum is taken over all balls in* $\mathbf{R}^n$.

Note that in general, the expression $\|f\|_*$ *will not be a norm but only a seminorm.*

If f *is a Schwartz function on* $\mathbf{R}^2$, *show that*

$$\|gf\|_* \leq C\|f\|_2.$$

Chapter 9

Singular multipliers

In this section, we establish estimates for an operator whose symbol is singular. The results we prove in this section are more involved than the simple L^2 multiplier theorem that we proved in Chapter 3. However, roughly speaking what we are doing is taking a singular symbol, smoothing it by convolution and then applying the L^2 multiplier theorem. As we shall see, this approach gives estimates in spaces of functions where we control the rate of increase near infinity. Estimates of this type were proven by Agmon and Hörmander. The details of our presentation are different, but the underlying ideas are the same.

9.1 Estimates for an operator with a singular symbol

For the next several chapters, we will be considering a differential operator in $\mathbf{R}^n$, $n \geq 3$,

$$\Delta + 2\zeta \cdot \nabla = e^{-x\cdot\zeta}\Delta e^{x\cdot\zeta}$$

where $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j \zeta_j = 0$.

Exercise 9.1 *Show that $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$ if and only if $\zeta = \xi + i\eta$ where ξ and η are in $\mathbf{R}^n$ and satisfy $|\xi| = |\eta|$ and $\xi \cdot \eta = 0$.*

Exercise 9.2 *a) Show that $\Delta e^{x\cdot\zeta} = 0$ if and only if $\zeta \cdot \zeta = 0$. b) Find conditions on $\tau \in \mathbf{R}$ and $\xi \in \mathbf{R}^n$ so that $e^{\tau t + x\cdot\xi}$ satisfies*

$$(\frac{\partial^2}{\partial t^2} - \Delta)e^{t\tau + x\cdot\xi} = 0.$$

The symbol of this operator is

$$-|\xi|^2 + 2i\zeta\cdot\xi = |\operatorname{Im}\zeta|^2 - |\operatorname{Im}\zeta + \xi|^2 + 2i\operatorname{Re}\zeta\cdot\xi.$$

Thus it is clear that this symbol vanishes on a sphere of codimension 2 which lies in the hyperplane $\operatorname{Re}\zeta\cdot\xi = 0$ and which has center $-\operatorname{Im}\zeta$ and radius $|\operatorname{Im}\zeta|$. Near this sphere, the symbol vanishes to first order. This means that the reciprocal of the symbol is locally integrable. In fact, we have the following fundamental estimate.

Lemma 9.3 *If $\eta \in \mathbf{R}^n$, $0 \le s \le 1$ and $r > 0$ then there exists a constant C depending only on the dimension n so that*

$$\int_{B_r(\eta)} \left| \frac{|\xi|^s}{-|\xi|^2 + 2i\zeta\cdot\xi} \right| d\xi \le \frac{Cr^{n-1}}{|\zeta|^{1-s}}.$$

Proof. We first observe that we are trying to prove a dilation invariant estimate, and we can simplify our work by scaling out one parameter. If we make the change of variables, $\xi = |\zeta| x$, we obtain

$$\int_{B_r(\eta)} \left| \frac{|\xi|^s}{-|\xi|^2 + 2i\zeta\cdot\xi} \right| d\xi = |\zeta|^{n-2+s} \int_{B_{r/|\zeta|}(\eta/|\zeta|)} \frac{|x|^s}{-|x|^2 + 2\hat{\zeta}\cdot x}\, dx$$

where $\hat{\zeta} = \zeta/|\zeta|$. Thus, it suffices to consider the estimate when $|\zeta| = 1$ and we assume below that we have $|\zeta| = 1$.

We also, may make rotation $\xi = Ox$ so that $O^t \operatorname{Re}\zeta = e_1/\sqrt{2}$, with e_1 the unit vector in the x_1 direction and $O^t \operatorname{Im}\zeta = e_2/\sqrt{2}$. Then, we have that

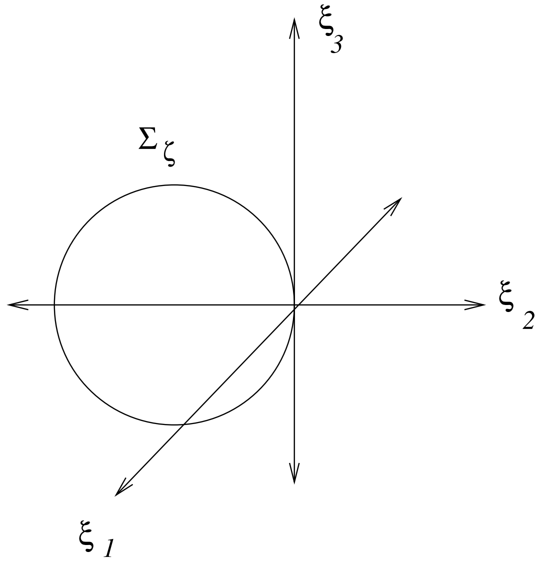
$$\int_{B_r(\eta)} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta\cdot\xi|}\, d\xi = \int_{B_r(O^t\eta)} \frac{|x|^s}{|-|x|^2 + 2iO^t\zeta\cdot x|}\, dx.$$

Thus, it suffices to prove the Lemma in the special case when $\zeta = (e_1 + ie_2)/\sqrt{2}$.

We let $\Sigma_\zeta = \{\xi : -|\xi|^2 + 2i\zeta\cdot\xi = 0\}$ be the zero set of the symbol.

Case 1. The ball $B_r(\eta)$ satisfies $r < 1/100$, $\operatorname{dist}(\eta, \Sigma_\zeta) < 2r$. Since $|\xi|^s$ is bounded by a constant on $B_r(\eta)$, it suffices to consider the case $s = 0$. We make an additional change of variables. We rotate in the variables $(\xi_2, \dots \xi_n)$ about the center of Σ_ζ, $-e_1/\sqrt{2}$, so that η is within $2r$ units of the origin. We can find a ball B_{3r} of radius $3r$ and centered 0 in Σ_ζ so that $B_r(\eta) \subset B_{3r}$ Now, we use coordinates $x_1 = \operatorname{Re}\zeta\cdot\xi$, $x_2 = |\operatorname{Im}\zeta|^2 - |\operatorname{Im}\zeta + \xi|^2$ and $x_j = \xi_j$ for $j = 3, \dots, n$. We leave it as an exercise to compute the Jacobian and show that it is bounded above and below on $B_r(\eta)$. This gives the bound

$$\int_{B_{3r}} \left| \frac{1}{-|\xi|^2 + 2i\zeta\cdot\xi} \right| d\xi \le C \int_{B_{Cr}(0)} \frac{1}{|x_1 + ix_2|}\, dx_1\, dx_2 \dots dx_n = Cr^{n-1}.$$



Case 2. We have $B_r(\eta)$ with $\text{dist}(\eta, \Sigma_\zeta) > 2r$. In this case, we have

$$\sup_{\xi \in B_r(\eta)} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \leq C/r.$$

If $\text{dist}(\eta, \Sigma_\zeta) < 8$, then this follows since $|-|\xi|^2 + 2i\zeta \cdot \xi|$ is comparable to $\text{dist}(\xi, \sigma_\zeta)$ on $B_r(\eta)$. If $\text{dist}(\eta, \Sigma_\zeta) \geq 8$, then $|-|\xi|^2 + 2i\zeta \cdot \xi|$ is comparable to $|\xi|^2$ for $\xi \in B_r(\eta)$. The Lemma follows easily.

Case 3. The ball $B_r(\eta)$ satisfies $r > 1/100$ and $\text{dist}(\eta, \Sigma_\zeta) < 2r$.

In this case, write $B_r(\eta) = B_0 \cup B_\infty$ where $B_0 = B_r(\eta) \cap B_4(0)$ and $B_\infty = B_r(\eta) \backslash B_4(0)$. By case 1 and 2,

$$\int_{B_0} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \, d\xi \leq C.$$

Since $B_4(0)$ contains the set Σ_ζ, one can show that

$$\frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \leq C/|\xi|^{2-s}$$

on B_∞ and integrating this estimate gives

$$\int_{B_\infty} \frac{|\xi|^s}{|-|\xi|^2 + 2i\zeta \cdot \xi|} \, d\xi \leq C r^{n-2+s}.$$

Since $r > 1/100$, the estimates on B_0 and B_∞ imply the estimate of the Lemma in this case. ∎

As a consequence of this Lemma, we can define the operator $G_\zeta : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by

$$G_\zeta f = \left(\frac{\hat{f}(\xi)}{-|\xi|^2 + 2i\xi \cdot \zeta} \right)^{\check{}}$$

Lemma 9.4 *The map G_ζ is bounded from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ and we have*

$$(\Delta + 2\zeta \cdot \nabla) G_\zeta f = G_\zeta (\Delta + 2\zeta \cdot \nabla) f = f$$

if $f \in \mathcal{S}(\mathbf{R}^n)$.

Proof. According to the previous lemma, the symbol of G_ζ satisfies the growth condition of Example 2.26 in Chapter 2. Hence $G_\zeta f$ is in $\mathcal{S}'(\mathbf{R}^n)$. The remaining results rely on the Proposition 1.18 of Chapter 1. ∎

It is not enough to know that $G_\zeta f$ is a tempered distribution. We would also like to know that the map G_ζ is bounded between some pair of Banach spaces. This will be useful when we try to construct solutions of perturbations of the operator $\Delta + 2\zeta \cdot \nabla$. The definition of the spaces we will use appears similar to the Besov spaces and the Littlewood-Paley theory in Chapter 7. However, now we are decomposing f rather than $\hat{f}$. To define these spaces, we let

$$B_j = B_{2^j}(0)$$

and then put $R_j = B_j \setminus B_{j-1}$. We let $\dot{M}^{p,s}_q(\mathbf{R}^n)$ denote the space of functions u for which the norm

$$\|u\|_{\dot{M}^{p,s}_q} = \left(\sum_{k=-\infty}^{\infty} \left[2^{ks} \|u\|_{L^p(R_k)} \right]^q \right)^{1/q} < \infty.$$

Also, we let $M^{p,s}_q$ be the space of measurable functions for which the norm

$$\|u\|_{M^{p,s}_q} = \left(\|u\|^q_{L^p(B_0)} + \sum_{k=1}^{\infty} \left[2^{ks} \|u\|_{L^p(R_k)} \right]^q \right)^{1/q}.$$

These definitions are valid for $0 < p \le \infty$, $s \in \mathbf{R}$ and $0 < q < \infty$. We will also need the case when $q = \infty$ and this is defined by replacing the ℓ^q norm of the sequence $2^{ks}\|u\|_{L^p(R_k)}$ by the supremum. Our primary interests are the spaces where $p = 2$, $q = 1$ and $s = 1/2$ and the space where $p = 2$, $q = \infty$ and $s = -1/2$. The following exercises give some practice with these spaces.

Exercise 9.5 *For which a do we have*

$$(1+|x|^2)^{a/2} \in M^{2,1/2}_\infty(\mathbf{R}^n).$$

Exercise 9.6 *Show that if $r \geq q$, then*

$$M^{2,s}_q \subset M^{2,s}_r.$$

Exercise 9.7 *Show that if $s > 0$, then*

$$M^{2,s}_1 \subset \dot{M}^{2,s}_1.$$

Exercise 9.8 *Let T be the multiplication operator*

$$Tf(x) = (1+|x|^2)^{-(1+\epsilon)/2} f(x).$$

Show that if $\epsilon > 0$, then

$$T : M^{2,-1/2}_\infty \to M^{2,1/2}_1.$$

Exercise 9.9 *Show that we have the inclusion $M^{2,1/2}_1 \subset L^2(\mathbf{R}^n, d\mu_s)$ where $d\mu_s = (1+|x|^2)^{2s}dx$. This means that we need to establish that for some C depending on n and s, we have the inequality*

$$\|u\|_{L^2(B_0)} + \sum_{k=1}^{\infty} \|u\|_{L^2(R_k)} \leq C\Big(\int_{\mathbf{R}^n} |u(x)|^2 (1+|x|^2)^s \, dx\Big)^{1/2}$$

Hint: The integral on $\mathbf{R}^n$ dominates the integral on each ring. On each ring, the weight changes by at most a fixed factor. Thus, it makes sense to replace the weight by its smallest value. This will give an estimate on each ring that can be summed to obtain the $M^{2,1/2}_1$ norm.

The main step of our estimate is the following lemma.

Lemma 9.10 *Let ψ and ψ' are Schwartz functions on $\mathbf{R}^n$ and set $\psi_k(x) = \psi(2^{-k}x)$ and $\psi'_j(x) = \psi'(2^{-j}x)$. We define a kernel $K : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$ by*

$$K(\xi_1, \xi_2) = \int_{\mathbf{R}^n} \frac{\hat{\psi}'_j(\xi_1 - \xi)\hat{\psi}_k(\xi - \xi_2)}{-|\xi|^2 + 2i\zeta \cdot \xi} \, d\xi.$$

Then there is a constant C *so that*

$$\sup_{\xi_1}\int |K(\xi_1,\xi_2)|\,d\xi_2 \;\le\; \frac{C2^j}{|\zeta|} \tag{9.11}$$

$$\sup_{\xi_2}\int |K(\xi_1,\xi_2)|\,d\xi_1 \;\le\; \frac{C2^k}{|\zeta|}. \tag{9.12}$$

As a consequence, the operator $T_{j,k}$ *given by*

$$T_{j,k}f(\xi_1) = \int K(\xi_1,\xi_2)f(\xi_2)\,d\xi_2$$

satisfies

$$\|T_{j,k}f\|_p \le \frac{C}{|\zeta|}2^{k/p}2^{j/p'}\|f\|_p. \tag{9.13}$$

Proof. Observe that $\hat\psi_k(\xi) = 2^{kn}\hat\psi(\xi 2^k) = (\hat\psi)_{2^{-k}}(\xi)$. Thus, $\|\hat\psi_k\|_1$ is independent of k. Since $\psi \in \mathcal{S}(\mathbf{R}^n)$, we have that $\|\hat\psi\|_1$ is finite. Thus if we use Tonelli's theorem, we have

$$\int_{\mathbf{R}^n} |K(\xi_1,\xi_2)|\,d\xi_2 \le \|\hat\psi\|_1 \int \frac{|\hat\psi_j'(\xi_1-\xi)|}{|-|\xi|^2+2i\zeta\cdot\xi|}\,d\xi.$$

To estimate the integral on the right of this inequality, we break the integral into rings centered at ξ_1 and use that $\hat\psi'$ decays rapidly at infinity so that, in particular, we have $\hat\psi'(\xi) \le C\min(1,|\xi|^{-n})$. Then applying Lemma 9.3 gives us

$$\begin{aligned}
\int \frac{|\hat\psi_j'(\xi_1-\xi)|}{|-|\xi|^2+2i\zeta\cdot\xi|}\,d\xi \;&\le\; \|\hat\psi\|_\infty 2^{nj}\int_{B_{2^{-j}}(\xi_1)} \frac{1}{|-|\xi|^2+2i\zeta\cdot\xi|}\,d\xi \\
&\quad + \sum_{l=1}^{\infty} C2^{nj}2^{-n(j-l)} \int_{B_{2^{-j+l}}(\xi_1)\setminus B_{2^{-j+l-1}}(\xi_1)} \frac{1}{|-|\xi|^2+2i\zeta\cdot\xi|}\,d\xi \\
&\le\; \frac{C}{|\zeta|}2^j\sum_{l=0}^{\infty}2^{-l}.
\end{aligned}$$

This gives the first estimate (9.11). The second is proven by interchanging the roles of ξ_1 and ξ_2. The estimate (9.11) gives a bound for the operator norm on L^∞. The estimate (9.12) gives a bound for the operator norm on L^1. The bound for the operator norm on L^p follows by the Riesz-Thorin interpolation theorem, Theorem 4.1. See exercise 4.5. ∎

Exercise 9.14 *Show that it suffices to prove the following theorem for* $|\zeta| = 1$. *That is, show that if the theorem holds when* $|\zeta| = 1$, *then by rescaling, we can deduce that the result holds for all* ζ *with* $\zeta \cdot \zeta = 0$.

Exercise 9.15 *The argument given should continue to prove an estimate as long as* $\operatorname{Re} \zeta$ *and* $\operatorname{Im} \zeta$ *are both nonzero. Verify this and show how the constants depend on* ζ.

Theorem 9.16 *The map* G_ζ *satisfies*

$$\sup_j 2^{-j/2} \|G_\zeta f\|_{L^2(B_j)} \leq \frac{C}{|\zeta|} \|f\|_{\dot{M}_1^{2,1/2}}$$

and

$$\sup_j 2^{-j/2} \|G_\zeta f\|_{L^2(B_j)} \leq \frac{C}{|\zeta|} \|f\|_{M_1^{2,1/2}}.$$

Proof. We first suppose that f is in the Schwartz space. We choose $\psi \geq 0$ as in Chapter 7 so that $\operatorname{supp} \psi \subset \{x : 1/2 \leq x \leq 2\}$ and with $\psi_k(x) = \psi(2^{-k}x)$, we have

$$\sum_{k=-\infty}^{\infty} \psi_k^2 = 1, \qquad \text{in } \mathbf{R}^n \setminus \{0\}.$$

We let $\phi = 1$ if $|x| < 1$, $\phi \geq 0$, $\phi \in \mathcal{D}(\mathbf{R}^n)$ and set $\phi_j(x) = \phi(2^{-j}x)$. We decompose f using the ψ_k's to obtain

$$\phi_j G_\zeta f = \sum_{k=-\infty}^{\infty} \phi_j G_\zeta \psi_k^2 f.$$

The Plancherel theorem implies that

$$\|\phi_j G_\zeta(\psi_k^2 f)\|_2^2 = (2\pi)^{-n} \int |T_{j,k} \widehat{\psi_k f}|^2 \, d\xi.$$

Here, the operator $T_{j,k}$ is as in the previous lemma but with ψ replaced by ϕ and ψ' replaced by ψ. Hence, from Lemma 9.10 we can conclude that

$$\|\phi_j G_\zeta \psi_k^2 f\|_2 \leq \frac{C}{|\zeta|} 2^{j/2} 2^{k/2} \sum_{|\ell| \leq 1} \|\psi_{k+\ell} f\|_2. \tag{9.17}$$

Now, using Minkowski's inequality, we have

$$\|G_\zeta f\|_{L^2(B_j)} \leq C \sum_{k=-\infty\infty} \|\phi_j G_\zeta \psi_k^2 f\|_{L^2(\mathbf{R}^n)}. \tag{9.18}$$

The first conclusion of the theorem now follows from (9.17) and (9.18).

The estimate in the inhomogeneous space follows by using Cauchy-Schwarz to show

$$\sum_{k=-\infty}^{0} 2^{k/2}\|f\|_{L^2(R_k)} \leq \left(\sum_{k=-\infty}^{0} \|f\|^2_{L^2(R_k)}\right)^{1/2} \left(\sum_{k=-\infty}^{0} 2^{k/2}\right)^{1/2} = \left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^{1/2} \|f\|_{L^2(B_0)}.$$

Finally, to remove the restriction that f is in the Schwartz space, we observe that the Lemma below tells us that Schwartz functions are dense in $\dot{M}^{2,1/2}_1$ and $M^{2,1/2}_1$. ∎

Lemma 9.19 *We have that $\mathcal{S}(\mathbf{R}^n)\cap \dot{M}^{2,1/2}_1$ is dense in $\dot{M}^{2,1/2}_1$ and $\mathcal{S}(\mathbf{R}^n)\cap M^{2,1/2}_1$ is dense in $\dot{M}^{2,1/2}_1$.*

Proof. To see this, first observe that if we pick f in $\dot{M}^{2,1/2}_1$ and define

$$f_N(x) = \begin{cases} 0, & |x| < 2^{-N} \text{ or } |x| > 2^N \\ f(x), & 2^{-N} \leq |x| \leq 2^N \end{cases}$$

then f_N converges to$f$ in $\dot{M}^{2,1/2}_1$. Next, if we regularize with a standard mollifier, then $f_{N,\epsilon} = f_N * \eta_\epsilon$ converges to f_N in L^2. If we assume that η is supported in the unit ball, then for $\epsilon < 2^{-N-1}$, $f_{N,\epsilon}$ will be supported in the shell $\{x : 2^{-N-1} \leq |x| \leq 2^{N+1}\}$. For such functions, we may use Cauchy-Schwarz to obtain

$$\|f_N - f_{N,\epsilon}\|_{\dot{M}^{2,1}_1} \leq \left(\sum_{k=-N}^{N+1} \|f_{N,\epsilon} - f_N\|^2_{L^2(R_k)}\right)^{1/2} \left(\sum_{k=-N}^{N+1} 2^k\right)^{1/2} = C\|f_N - f_{N,\epsilon}\|_2.$$

Hence, for functions supported in compact subsets of $\mathbf{R}^n \setminus \{0\}$, the L^2 convergence of $f_{N,\epsilon}$ to f_N implies convergence in the space $\dot{M}^{2,1/2}_1$. Approximation in $M^{2,1/2}_1$ is easier since we only need to cut off near infinity. ∎

Exercise 9.20 *Are Schwartz functions dense in $M^{2,-1/2}_\infty$?*

Exercise 9.21 *Use the ideas above to show that*

$$\sup_j 2^{-j/2}\|\nabla G_\varsigma f\|_{L^2(B_j)} \leq C\|f\|_{\dot{M}^{2,1/2}_1}.$$

Hint: One only needs to find a replacement for Lemma 9.3.

Exercise 9.22 *Use the ideas above to show that $I_\alpha : \dot{M}^{2,\alpha/2}_1 \to \dot{M}^{2,-\alpha/2}_\infty$. Hint: Again, the main step is to find a substitute for Lemma 9.3.*

Finally, we establish uniqueness for the equation $\Delta u + 2\zeta \cdot \nabla u = 0$ in all of $\mathbf{R}^n$. In order to obtain uniqueness, we will need some restriction on the growth of u at infinity.

Theorem 9.23 *If u in L^2_{loc} and satisfies*

$$\lim_{j\to\infty} 2^{-j}\|u\|_{L^2(B_j(0))} = 0$$

and $\Delta u + 2\zeta \cdot \nabla u = 0$, then $u = 0$.

The following is taken from Hörmander [18], see Theorem 7.1.27.

Lemma 9.24 *If u is a tempered distribution which satisfies*

$$\limsup_{R\to\infty} R^{-d}\|u\|_{L^2(B_R(0))} = M < \infty$$

and $\hat{u}$ is supported in a compact surface S of codimension d, then there is a function $u_0 \in L^2(S)$ so that

$$\hat{u}(\phi) = \int_S \phi u_0 d\sigma$$

and $\|u_0\|_{L^2(S)} \le CM$.

Proof. We choose $\phi \in \mathcal{D}(\mathbf{R}^n)$, $\operatorname{supp}\phi \subset B_1(0)$, ϕ even, $\int \phi = 1$ and consider $\hat{u} * \phi_\epsilon$. By Plancherel's theorem, we have that

$$\int |\hat{u} * \phi_{2^{-j}}|^2 \, d\xi = \int |\hat{\phi}(2^{-j}x)u(x)|^2 \, dx \le C2^{-dj}M^2.$$

To establish this, we break the integral into the integral over the unit ball and integrals over shells. We use that $\hat{\phi}$ is in $\mathcal{S}(\mathbf{R}^n)$ and satisfies $|\hat{\phi}(x)| \le C\min(1, |x|^{-(d+1)})$. For j large enough so that $2^{-jd}\|u\|_{L^2(B_j)} \le 2M$, we have

$$\begin{aligned}\int |\hat{\phi}(2^{-j}x)|^2|u(x)|^2 \, dx &\le \int_{B_j} |u(x)|^2 \, dx + \sum_{k=j}^{\infty} \int_{R_{k+1}} |\hat{\phi}(2^{-j}x)|^2|u(x)|^2 \, dx \\ &\le C2^{2jd}4M^2 + C2^{2j(d+1)} \sum_{k=j}^{\infty} 2^{-2k}4M^2 = CM^2 2^{dj}.\end{aligned}$$

If we let $S_\epsilon = \{\xi : \operatorname{dist}(\xi, \operatorname{supp} S) < \epsilon\}$ and ψ is in the Schwartz class, then we have

$$\int_S |\psi(x)|^2 \, d\sigma = C_d \lim_{\epsilon\to 0^+} \epsilon^{-d} \int_{S_\epsilon} |\psi(x)|^2 \, dx.$$

Since $\phi_\epsilon * \psi \to \psi$ in $\mathcal{S}$, we have

$$\hat{u}(\psi) = \lim_{j\to\infty} \hat{u}(\phi_{2^{-j}} * \psi).$$

Then using Cauchy-Schwarz and the estimate above for $u * \phi_{2^{-j}}$, we obtain

$$|\hat{u}(\psi * \phi_{2^{-j}})| = |\int_{S_{2^{-j}}} \hat{u} * \phi_{2^{-j}}(x)\psi(x)\,dx| \le CM2^{jd}(\int_{S_{2^{-j}}} |\psi(x)|^2\,dx)^{1/2}.$$

If we let $\epsilon \to 0^+$, we obtain that $|\hat{u}(\psi)| \le CM\|\psi\|_{L^2(S)}$. This inequality implies the existence of u_0. ∎

Now we can present the proof of our uniqueness theorem.

Proof of Theorem 9.23. Since $\Delta u + 2\zeta\cdot\nabla u = 0$, we can conclude that the distribution $\hat{u}$ is supported on the zero set of $-|\xi| + 2i\zeta\cdot\xi$, a sphere of codimension 2. Now the hypothesis on the growth of the L^2 norm and the previous lemma, Lemma 9.24 imply that $\hat{u} = 0$. ∎

Corollary 9.25 *If f is in $M_1^{2,1/2}$, then there is exactly one solution u of*

$$\Delta u + 2\zeta\cdot\nabla u = f$$

which lies in $M_\infty^{2,-1/2}$. This solution satisfies

$$|\zeta|\|u\|_{M_\infty^{2,-1/2}} + \|\nabla u\|_{M_\infty^{2,-1/2}} \le C\|f\|_{M_1^{2,1/2}}.$$

Proof. The existence follows from Theorem 9.16 and exercise 9.21. If u is in $M_\infty^{2,-1/2}$, then we have u is in L^2_{loc} and that

$$\lim_{j\to\infty} 2^{-\alpha j}\|u\|_{L^2(B_{2^j}(0))} = 0$$

if $\alpha > 1/2$. Thus, the uniqueness follows from Theorem 9.23. ∎

9.2 A trace theorem.

The goal of this section is to provide another application of the ideas presented above. The result proven will not be used in this course. Also, this argument will serve to introduce a technical tool that will be needed in Chapter 15.

We begin with a definition of a *Ahlfors condition.* We say that a Borel measure μ in $\mathbf{R}^n$ satisfies an Ahlfors condition if for some constant A it satisfies $\mu(B_r(x)) \leq Ar^{n-1}$. This is a property which is satisfied by surface measure on the boundary of a C^1-domain as well as by surface measure on a graph $\{(x', x_n) : x_n = \phi(x')\}$ provided that the $\|\nabla\phi\|_\infty < \infty$.

Our main result is the following theorem.

Theorem 9.26 *If f is in $\mathcal{S}(\mathbf{R}^n)$ and μ satisfies the Ahlfors condition, then there is a constant C so that*

$$\int_{\mathbf{R}^n} |\hat{u}(\xi)|^2 \, d\mu \leq C\|u\|^2_{\dot{M}_1^{2,1/2}}.$$

This may seem peculiar, but as an application, we observe that this theorem implies a trace theorem for Sobolev spaces.

Corollary 9.27 *If μ satisfies the Ahlfors condition and $s > 1/2$ then we have*

$$\int_{\mathbf{R}^n} |u|^2 \, d\mu \leq C\|u\|^2_{L^2_s(\mathbf{R}^n)}.$$

Proof. First assume that $u \in \mathcal{S}(\mathbf{R}^n)$. Applying the previous theorem to $\check{u}(x) = (2\pi)^{-n}\hat{u}(-x)$ gives that

$$\int |u|^2 \, d\mu(x) \leq C\|\hat{u}\|_{\dot{M}_1^{2,1/2}}.$$

It is elementary (see exercise 9.9), to establish the inequality

$$\|v\|_{M_1^{2,1/2}} \leq C_s \int_{\mathbf{R}^n} |v(x)|^2(1+|x|^2)^s \, dx$$

when $s > 1/2$. Also, from exercise 9.7 or the proof of theorem 9.16, we have

$$\|v\|_{\dot{M}_1^{2,1/2}} \leq C_s\|v\|_{M_1^{2,1/2}}.$$

Combining the two previous inequalities with $v = \hat{u}$ gives the desired conclusion. ∎

Lemma 9.28 *The map $g \to \int \cdot g\, dx$ is an isomorphism from $\dot{M}^{2,-1/2}_\infty$ to the dual space of $M^{2,1/2}_1$, $\dot{M}^{2,1/2}_1{}'$.*

Proof. It is clear by applying Hölder's inequality twice that

$$\int_{\mathbf{R}^n} fg\, dx \le \|f\|_{\dot{M}^{2,1/2}_1} \|g\|_{M^{2,-1/2}_\infty}.$$

Thus, our map takes $\dot{M}^{2,1/2}_\infty$ into the dual of $\dot{M}^{2,1/2}_1$. To see that this map is onto, suppose that $\lambda \in M^{2,1/2'}_1$. Observe $L^2(R_k) \subset \dot{M}^{2,1/2}_1$ in the sense that if $f \in L^2(R_k)$, then the function which is f in R_k and 0 outside R_k lies in $\dot{M}^{2,1/2}_1$. Thus, for such f,

$$\lambda(f) \le \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \|f\|_{\dot{M}^{2,1/2}_\infty} = 2^{k/2} \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \|f\|_{L^2(R_k)}.$$

Since we know the dual of $L^2(R_k)$, we can conclude that there exists g_k with

$$\|g_k\|_{L^2(R_k)} \le 2^{k/2} \|\lambda\|_{\dot{M}^{2,1/2}_1{}'} \tag{9.29}$$

so that

$$\lambda(f) = \int_{R_k} fg\, dx \tag{9.30}$$

for $f \in L^2(R_k)$. We set $g = \sum_{k=-\infty}^{\infty} g_k$. Note that there can be no question about the meaning of the infinite sum since for each x at most one summand is not zero. The estimate (9.29) implies $\|g\|_{\dot{M}^{2,-1/2}_\infty} \le \|\lambda\|_{\dot{M}^{2,1/2}_1{}'}$. If f is supported in $\cup_{k=-N}^{N} R_k$, then summing (9.30) implies that

$$\lambda(f) = \int fg\, dx.$$

Finally, such f are dense in $\dot{M}^{2,1/2}_1$, so we conclude $\lambda(f) = \int fg\, dx$ for all f. ∎

We have defined the adjoint of an operator on a Hilbert space in exercise 6.11. Here, we need a slightly more general notion. If $T : X \to \mathcal{H}$ is a continuous linear map from a normed vector space into a Hilbert space, then $x \to \langle Tx, y\rangle$ is a continuous linear functional of X. Thus, there exists $y* \in X'$ so that $y^*(x) = \langle Tx, y\rangle$. One can show that the map $y \to y^* = T^*y$ is linear and continuous. The map $T^* : \mathcal{H} \to X'$ is the *adjoint* of the map T. There adjoint discussed here is closely related to the transpose of a map introduced when we discussed distributions. For our purposes, the key distinction is that the transpose satisfies $(Tf, g) = (f, T^t g)$ for a bilinear pairing, while the adjoint is satisfies $\langle Tf, g\rangle = \langle f, T^* g\rangle$ for a sesquilinear pairing (this means linear in first variable and conjugate linear in the second variable). The map $T \to T^t$ will be linear, while the map $T \to T^*$ is conjugate linear.

The following lemma is a simple case of what is known to harmonic analysts as the Peter Tomás trick. It was used to prove a restriction theorem for the Fourier transform in [38].

Lemma 9.31 *Let $T : X \to \mathcal{H}$ be a map from a normed vector space X into a Hilbert space $\mathcal{H}$. If $T^*T : X \to X'$, and*

$$\|T^*Tf\|_{X'} \le A^2 \|f\|_X$$

then

$$\|Tf\|_{\mathcal{H}} \le A\|f\|_X.$$

Proof. We have

$$T^*Tf(f) = \langle Tf, Tf\rangle = \|Tf\|^2_{\mathcal{H}}$$

and since $|T^*Tf(f)| \le \|T^*Tf\|_{X'}\|f\|_X \le A^2\|f\|_X$, the lemma follows. ∎

Proof of Theorem 9.26. We consider f in $\dot{M}^{2,1/2}_1$ and let T denote the map $f \to \hat{f}$ as a map into $L^2(\mu)$. The map T^*T is given by

$$T^*Tf(x) = \int_{\mathbf{R}^n} \hat{f}(\xi)e^{-ix\cdot\xi}\, d\mu(\xi).$$

Using the Ahlfors condition on the measure μ one may repeat word for word our proof of Theorem 9.16 to conclude T^*T maps $\dot{M}^{2,1/2}_1 \to \dot{M}^{2,-1/2}_\infty$. Now the two previous Lemmas give that $T : \dot{M}^{2,1/2}_1 \to L^2(\mu)$. ∎

Exercise 9.32 *Prove a similar result for other co-dimensions–even fractional ones. That is suppose that $\mu(B_r(x)) \le Cr^{n-\alpha}$ for $0 < \alpha < n$. Then show that*

$$\int_{\mathbf{R}^n} |\hat{f}(\xi)|^2\, d\mu(\xi) \le C\|f\|_{\dot{M}^{2,\alpha/2}_1}.$$

Chapter 10

The Dirichlet problem for elliptic equations.

In this chapter, we introduce some of the machinery of elliptic partial differential equations. This will be needed in the next chapter to introduce the inverse boundary value problem we will study.

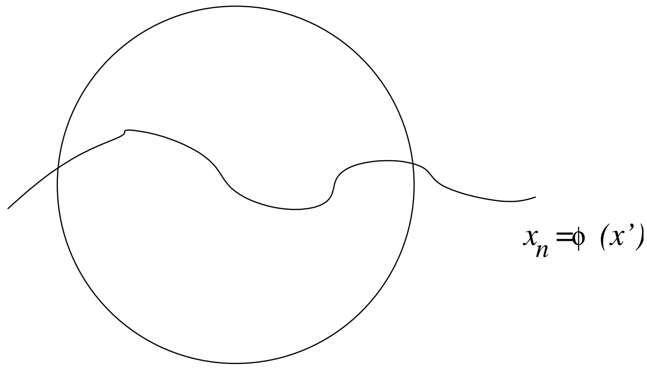
10.1 Domains in $\mathbf{R}^n$

For $\mathcal{O}$ an open subset of $\mathbf{R}^n$, we let $C^k(\mathcal{O})$ denote the space of functions on $\mathcal{O}$ which have continuous partial derivatives of all orders α with $|\alpha| \leq k$. We let $C^k(\bar{\mathcal{O}})$ be the space of functions for which all derivatives of order up to k extend continuously to the closure $\mathcal{O}$, $\bar{\mathcal{O}}$. Finally, we will let $\mathcal{D}(\mathcal{O})$ to denote the space of functions which are infinitely differentiable and are compactly supported in $\mathcal{O}$.

We say that $\Omega \subset \mathbf{R}^n$ is a *domain* if Ω is a bounded connected open set. We say that a domain is of class C^k if for each $x \in \Omega$, there is an $r > 0$, $\phi \in C^k(\mathbf{R}^{n-1})$ and coordinates $(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ (which we assume are a rotation of the standard coordinates) so that

$$\begin{aligned} \partial\Omega \cap B_{2r}(x) &= \{(x', x_n) : x_n = \phi(x')\} \\ \Omega \cap B_{2r}(x) &= \{(x', x_n) : x_n > \phi(x')\}. \end{aligned}$$

Here, $\partial\Omega$ is the boundary of a set. We will need that the map $x \to (x', 2\phi(x') - x_n)$ map $\Omega \cap B_r(x)$ into $\bar{\Omega}^c$. This can always be arranged by decreasing r. We also will assume that $\nabla\phi$ is bounded in all of $\mathbf{R}^{n-1}$.



In these coordinates, we can define *surface measure* $d\sigma$ on the boundary by

$$\int_{B_r(x)\cap\partial\Omega} f(y)\, d\sigma(y) = \int_{B_r(x)\cap\{y : y_n=\phi(y')\}} f(y', \phi(y'))\sqrt{1+|\nabla\phi(y')|^2}\, dy'.$$

Also, the vector field $\nu(y) = (\nabla\phi(y'), -1)(1+|\nabla\phi(y')|^2)^{-1/2}$ defines a unit outer normal for $y \in B_r(x) \cap \partial\Omega$.

Since our domain is bounded, the boundary of Ω is a bounded, closed set and hence compact. Thus, we may always find a finite collection of balls, $\{B_r(x_i) : i = 1, \dots, N\}$ as above which cover $\partial\Omega$.

Many of arguments will proceed more smoothly if we can divide the problem into pieces, choose a convenient coordinate system for each piece and then make our calculations in this coordinate system. To carry out these arguments, we will need partitions of unity. Given a collection of sets, $\{A_\alpha\}$, which are subsets of a topological space X, a partition of unity subordinate to $\{A_\alpha\}$ is a collection of real-valued functions $\{\phi_\alpha\}$ so that $\operatorname{supp}\phi_\alpha \subset A_\alpha$ and so that $\sum_\alpha \phi_\alpha = 1$. Partitions of unity are used to take a problem and divide it into bits that can be more easily solved. For our purposes, the following will be useful.

Lemma 10.1 *If K is a compact subset in $\mathbf{R}^n$ and $\{U_1, \dots, U_N\}$ is a collection of open sets which cover K, then we can find a collection of functions ϕ_j with each ϕ_j in $\mathcal{D}(U_j)$, $0 \le \phi_j \le 1$ and $\sum_{j=1}^N \phi_j = 1$ on K.*

Proof. By compactness, we can find a finite collection of balls $\{B_k\}_{k=1}^M$ so that each $\bar{B}_k$ lies in some U_j and the balls cover K. If we let $\mathcal{F} = \cup\bar{B}_k$ be the union of the closures of the balls B_k, then the distance between K and $\mathbf{R}^n \setminus \mathcal{F}$ is positive. Hence, we can find finitely many more balls $\{B_{M+1}, \dots, B_{M+L}\}$ to our collection which cover ∂F and which are contained in $\mathbf{R}^n \setminus K$. We now let $\tilde{\eta}_k$ be the standard bump translated and rescaled to the ball B_k. Thus if $B_k = B_r(x)$, then $\tilde{\eta}_k(y) = \exp(-1/(r^2 - |y-x|^2))$ in B_k and 0 outside B_k. Finally, we put $\tilde{\eta} = \sum_{k=1}^{M+L} \tilde{\eta}_k$ and then $\eta_k = \tilde{\eta}_k/\tilde{\eta}$. Each η_k, $k = 1, \ldots, M$ is smooth since $\tilde{\eta}$ is strictly positive on $\mathcal{O}$. Then we have $\sum_{k=1}^{M} \eta_k = 1$ on K and we may group to obtain one ϕ_j for each U_j. ∎

The following important result is the Gauss divergence theorem. Recall that for a $\mathbf{C}^n$ valued function $F = (F_1, \ldots, F_n)$, the *divergence* of F, is defined by

$$\operatorname{div} F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.$$

Theorem 10.2 *(Gauss divergence theorem) Let Ω be a C^1 domain and let $F : \Omega \to \mathbf{C}^n$ be in $C^1(\bar{\Omega})$. We have*

$$\int_{\partial\Omega} F(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega} \operatorname{div} F(x)\, dx.$$

The importance of this result may be gauged by the following observation: the theory of weak solutions of elliptic pde (and much of distribution theory) relies on making this result an axiom.

An important Corollary is the following version of Green's identity. In this Corollary and below, we should visualize the gradient of u, ∇u as a column vector so that the product $A\nabla u$ is makes sense as a matrix product.

Corollary 10.3 *If Ω is a C^1-domain, v is in $C^1(\bar{\Omega})$, u is in $C^2(\bar{\Omega})$ and $A(x)$ is an $n \times n$ matrix with $C^1(\bar{\Omega})$ entries, then*

$$\int_{\partial\Omega} v(x) A(x) \nabla u(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega} A(x)\nabla u(x) \cdot \nabla v(x) + v(x) \operatorname{div} A(x) \nabla u(x)\, dx.$$

Proof. Apply the divergence theorem to $vA\nabla u$. ∎

Next, we define *Sobolev spaces* on open subsets of $\mathbf{R}^n$. Our definition is motivated by the result in Proposition 3.13. For k a positive integer, we say that $u \in L^{2,k}(\Omega)$ if u has weak or distributional derivatives for all α for $|\alpha| \leq k$ and these derivatives, $\partial^\alpha u/\partial x^\alpha$, lie in $L^2(\Omega)$. This means that for all test functions $\phi \in \mathcal{D}(\Omega)$, we have

$$\int_{\Omega} u \frac{\partial^\alpha}{\partial x^\alpha} \phi(x)\, dx = (-1)^{|\alpha|} \int_{\Omega} \phi \frac{\partial^\alpha}{\partial x^\alpha} u(x)\, dx.$$

The weak derivatives of u are defined as we defined the derivatives of a tempered distribution. The differences are that since we are on a bounded open set, our functions are supported there and in this instance we require that the derivative be a distribution given by a function.

It should be clear how to define the norm in this space. In fact, we have that these spaces are Hilbert spaces with the inner product defined by

$$\langle u, v\rangle = \int_\Omega \sum_{|\alpha|\leq k} \frac{\partial^\alpha u}{\partial x^\alpha}\frac{\partial^\alpha \bar{v}}{\partial x^\alpha}\, dx. \tag{10.4}$$

We let $\|u\|_{L^{2,k}(\Omega)}$ be the corresponding norm.

Exercise 10.5 *Show that if Ω is a bounded open set, then $C^k(\bar{\Omega}) \subset L^{2,k}(\Omega)$.*

Example 10.6 *If u is in the Sobolev space $L^{2,k}(\mathbf{R}^n)$ defined in Chapter 3 and Ω is an open set, then the restriction of u to Ω, call it ru, is in the Sobolev space $L^{2,k}(\Omega)$. If Ω has reasonable boundary, (C^1 will do) then the restriction map $r : L^{2,k}(\mathbf{R}^n) \to L^{2,k}(\Omega)$ is onto. However, this may fail in general.*

Exercise 10.7 *a) Prove the product rule for weak derivatives. If ϕ is in $C^k(\bar{\Omega})$ and all the derivatives of ϕ, $\partial^\alpha\phi/\partial x^\alpha$ with $|\alpha| \leq k$ are bounded, then we have that*

$$\frac{\partial^\alpha \phi u}{\partial x^\alpha} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\frac{\partial^\beta \phi}{\partial x^\beta}\frac{\partial^\gamma u}{\partial x^\gamma}.$$

b)If $\phi \in C^k(\bar{\Omega})$, conclude that the map $u \to \phi u$ takes $L^{2,k}(\Omega)$ to $L^{2,k}(\Omega)$ and is bounded.

c) If $\phi \in C^1(\bar{\Omega})$, show that the map $u \to \phi u$ maps $L^{2,1}_0(\Omega) \to L^{2,1}_0(\Omega)$.

Lemma 10.8 *If Ω is a C^1 domain and u is in the Sobolev space $L^{2,k}(\Omega)$, then we may write $u = \sum_{j=0}^N u_j$ where u_0 has support in a fixed (independent of u) compact subset of Ω and each u_j, $j = 1, \ldots, N$ is supported in a ball $B_{r_j}(x_j)$ as in the definition of C^1 domain.*

Proof. We cover the boundary, $\partial\Omega$ by balls $\{B_1, \ldots, B_N\}$ as in the definition of C^1 domain. Then, $K = \Omega \setminus \cup_{k=1}^N B_k$ is a compact set so that the distance from K to $\mathbf{R}^n \setminus \Omega$ is positive, call this distance δ. Thus, we can find an open set $U_0 = \{x : \mathrm{dist}(x, \partial\Omega) > \delta/2\}$ which contains K and is a positive distance from $\partial\Omega$. We use Lemma 10.1 to make a partition of unity $1 = \sum_{j=0}^N \eta_j$ for the open cover of $\bar{\Omega}$ $\{U_0, B_1 \ldots, B_N\}$ and then we decompose $u = \sum_{j=0}^N \eta_j u$. The product rule of exercise 10.7 allows us to conclude that each term $u_j = \eta_j u$ is in $L^{2,k}(\Omega)$. ∎

Recall that we proved in Chapter 2 that smooth (Schwartz, actually) functions are dense in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$. One step of the argument involved considering the map

$$\eta * u$$

where η is a Schwartz function with $\int \eta = 1$. This approach may appear to break down u is only defined in an open subset of $\mathbf{R}^n$, rather than all of $\mathbf{R}^n$. However, we can make sense of the convolution in most of Ω if we require that the function η have compact support. Thus, we let $\eta \in \mathcal{D}(\mathbf{R}^n)$ be supported in $B_1(0)$ and have $\int \eta = 1$.

Lemma 10.9 *Suppose u is the Sobolev space $L^{p,k}(\Omega)$, $1 \le p < \infty$, for $k = 0, 1, 2, \dots$. Set $\Omega_\epsilon = \Omega \cap \{x : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. If we set $u_\delta = \eta_\delta * u$, then for $|\alpha| \le k$, we have*

$$\frac{\partial^\alpha}{\partial x^\alpha} u_\delta = (\frac{\partial^\alpha}{\partial x^\alpha} u)_\delta, \qquad \text{for } x \in \Omega_\epsilon \text{ with } \delta < \epsilon.$$

Hence, for each $\epsilon > 0$, we have

$$\lim_{\delta \to 0^+} \|u - u_\delta\|_{L^{p,k}(\Omega_\epsilon)} = 0.$$

Proof. We assume that u is defined to be zero outside of Ω. The convolution $u * \eta_\delta(x)$ is smooth in all of $\mathbf{R}^n$ and we may differentiate inside the integral and then express the x derivatives as y derivatives to find

$$\frac{\partial^\alpha}{\partial x^\alpha} u * \eta_\delta(x) = \int_\Omega u(y) \frac{\partial^\alpha}{\partial x^\alpha} \eta_\delta(x-y)\, dy = (-1)^{|\alpha|} \int_\Omega u(y) \frac{\partial^\alpha}{\partial y^\alpha} \eta_\delta(x-y)\, dy.$$

If we have $\delta < \epsilon$ and $x \in \Omega_\epsilon$, then $\eta_\delta(x - \cdot)$ will be in the space of test functions $\mathcal{D}(\Omega)$. Thus, we can apply the definition of weak derivative to conclude

$$(-1)^{|\alpha|} \int_\Omega u(y) \frac{\partial^\alpha}{\partial y^\alpha} \eta_\delta(x-y)\, dy = \int_\Omega (\frac{\partial^\alpha}{\partial y^\alpha} u(y)) \eta_\delta(x-y)\, dy.$$

∎

Lemma 10.10 *If Ω is a C^1-domain and $k = 0, 1, 2, \dots$, then $C^\infty(\bar\Omega)$ is dense in $L^{2,k}(\Omega)$.*

Proof. We may use Lemma 10.8 to reduce to the case when u is zero outside $B_r(x) \cap \Omega$ for some ball centered at a boundary point x and $\partial\Omega$ is given as a graph, $\{(x', x_n) : x_n = \phi(x')\}$ in $B_r(x)$. We may translate u to obtain $u_\epsilon(x) = u(x + \epsilon e_n)$. Since u_ϵ has weak derivatives in a neighborhood of Ω, by the local approximation lemma, Lemma 10.9 we may approximate each u_ϵ by functions which are smooth up to the boundary of Ω. ∎

Lemma 10.11 *If Ω and Ω' are bounded open sets and $F : \Omega \to \Omega'$ is $C^1(\bar{\Omega})$ and $F^{-1} : \Omega' \to \Omega$ is also $C^1(\bar{\Omega}')$, then we have $u \in L^{2,1}(\Omega')$ if and only if $u \circ F \in L^{2,1}(\Omega)$.*

Proof. The result is true for smooth functions by the chain rule and the change of variables formulas of vector calculus. Note that our hypothesis that F is invertible implies that the Jacobian is bounded above and below. The density result of Lemma 10.9 allows us to extend to the Sobolev space. ∎

Lemma 10.12 *If Ω is a C^1-domain, then there exists an extension operator $E : L^{2,k}(\Omega) \to L^{2,k}(\mathbf{R}^n)$.*

Proof. We sketch a proof when $k = 1$. We will not use the more general result. The general result requires a more substantial proof. See the book of Stein [29], whose result has the remarkable feature that the extension operator is independent of k.

For the case $k = 1$, we may use a partition of unity and to reduce to the case where u is nonzero outside $B_r(x) \cap \Omega$ and that $\partial\Omega$ is the graph $\{(x', x_n) : x_n = \phi(x')\}$ in $B_r(x)$. By the density result, Lemma 10.10, we may assume that u is smooth up to the boundary. Then we can define Eu by

$$Eu(x) = \begin{cases} u(x), & x_n > \phi(x') \\ u(x', 2\phi(x') - x_n), & x_n < \phi(x') \end{cases}$$

If ψ is test function in $\mathbf{R}^n$, then we can apply the divergence theorem in Ω and in $\mathbf{R}^n \setminus \bar{\Omega}$ to obtain that

$$\begin{aligned} \int_\Omega Eu\frac{\partial\psi}{\partial x_j} + \psi\frac{\partial Eu}{\partial x_j}\,dx &= \int_{\partial\Omega} \psi Eu\nu \cdot e_j\,d\sigma \\ \int_{\mathbf{R}^n\setminus\bar{\Omega}} Eu\frac{\partial\psi}{\partial x_j} + \psi\frac{\partial Eu}{\partial x_j}\,dx &= -\int_{\partial\Omega} \psi Eu\nu \cdot e_j\,d\sigma \end{aligned}$$

In the above expressions, the difference in sign is due to the fact that the normal which points out of Ω is the negative of the normal which points out of $\mathbf{R}^n \setminus \bar{\Omega}$.

Adding these two expressions, we see that Eu has weak derivatives in $\mathbf{R}^n$. These weak derivatives are given by the (ordinary) derivative $\partial Eu/\partial x_j$, which is defined except on $\partial\Omega$. In general, Eu will not have an ordinary derivative on $\partial\Omega$. Using Lemma 10.11, one can see that this extension operator is bounded. The full extension operator is obtained by taking a function u, writing $u = \sum_{j=0}^N \eta_j u$ as in Lemma 10.8 where the support of η_0 does not meet the boundary. For each η_j which meets the boundary, we apply the local extension operator constructed above and then sum to obtain $Eu = \eta_0 u + \sum_{j=1}^N E(\eta_j u)$. Once we have defined the extension operator on smooth functions in $L^{2,1}$, then we can use the density result of Lemma 10.10 to define the extension operator on the full space. ▮

Next, we define an important subspace of $L^{2,1}(\Omega)$, $L^{2,1}_0(\Omega)$. This space is the closure of $\mathcal{D}(\Omega)$ in the norm of $L^{2,1}(\Omega)$. The functions in $L^{2,1}_0(\Omega)$ will be defined to be the Sobolev functions which vanish on the boundary. Since a function u in the Sobolev space is initially defined a.e., it should not be clear that we can define the restriction of u to a lower dimensional subset. However, we saw in Chapter 9 that this is possible. We shall present a second proof below. The space $L^{2,1}_0(\Omega)$ will be defined as the space of functions which have zero boundary values.

Remark: Some of you may be familiar with the spaces $L^{2,1}(\Omega)$ as $H^1(\Omega)$ and $L^{2,1}_0(\Omega)$ as $H^1_0(\Omega)$.

We define the boundary values of a function in $L^{2,1}(\Omega)$ in the following way. We say that $u = v$ on $\partial\Omega$ if $u - v \in L^{2,1}_0(\Omega)$. Next, we define a space $L^{2,1/2}(\partial\Omega)$ to be the equivalence classes $[u] = u + L^{2,1}_0(\Omega) = \{v : v - u \in L^{2,1}(\Omega)\}$. Of course, we need a norm to do analysis. The norm is given by

$$\|u\|_{L^{2,1/2}(\partial\Omega)} = \inf\{\|v\|_{L^{2,1}(\Omega)} : u - v \in L^{2,1}_0(\Omega)\}. \tag{10.13}$$

It is easy to see that this is a norm and the resulting space is a Banach space. It is less clear $L^{2,1/2}(\partial\Omega)$ is a Hilbert space. However, if the reader will recall the proof of the projection theorem in Hilbert space one may see that the space on the boundary, $L^{2,1/2}(\partial\Omega)$, can be identified with the orthogonal complement of $L^{2,1}_0(\Omega)$ in $L^{2,1}(\Omega)$ and thus inherits an inner product from $L^{2,1}(\Omega)$.

This way of defining functions on the boundary should be unsatisfyingly abstract to the analysts in the audience. The following result gives a concrete realization of the space.

Proposition 10.14 *Let Ω be a C^1-domain. The map*

$$r : C^1(\bar{\Omega}) \to L^2(\partial\Omega)$$

which takes ϕ to the restriction of ϕ to the boundary, $r\phi$ satisfies

$$\|ru\|_{L^2(\partial\Omega)} \leq C\|u\|_{L^{2,1}(\Omega)}$$

and as a consequence extends continuously to $L^{2,1}(\Omega)$. Since $r(L^{2,1}_0(\Omega)) = 0$, the map r is well-defined on equivalence classes in $L^{2,1/2}(\partial\Omega)$ and gives a continuous injection $r : L^{2,1/2}(\partial\Omega) \to L^2(\partial\Omega)$.

Exercise 10.15 *Prove the above proposition.*

Exercise 10.16 *If Ω is a C^1 domain, let H be a space of functions f on $\partial\Omega$ defined as follows. We say that $f \in H$ if for each ball $B_r(x)$ as in the definition of C^1 domains and each $\eta \in \mathcal{D}(B_r(x))$, we have $(\eta f)(y', \phi(y'))$ is in the space $L^{2,1/2}(\mathbf{R}^{n-1})$ defined in Chapter 3. In the above, ϕ is the function whose graph describes the boundary of Ω near x. A norm in the space H may be defined by fixing a covering of the boundary by balls as in the definition of C^1-domains, and then a partition of unity subordinate to this collection of balls, $\sum \eta_k$ and finally taking the sum*

$$\sum_k \|\eta_k f\|_{L^{2,1/2}(\mathbf{R}^{n-1})}$$

.

Show that $H = L^{2,1/2}(\partial\Omega)$.

Hint: We do not have the tools to solve this problem. Thus this exercise is an excuse to indicate the connection without providing proofs.

Lemma 10.17 *If Ω is a C^1 domain and $u \in C^1(\bar{\Omega})$, then there is a constant C so that*

$$\int_{\partial\Omega} |u(x)|^2 \, d\sigma(x) \le C \int_{\Omega} |u(x)|^2 + |\nabla u(x)|^2 \, dx.$$

Proof. According to the definition of a C^1-domain, we can find a finite collection of balls $\{B_j : j = 1, \ldots, N\}$ and in each of these balls, a unit vector, α_j, which satisfies $\alpha_j \cdot \nu \ge \delta > 0$ for some constant δ. To do this, choose α_j to be $-e_n$ in the coordinate system which is used to describe the boundary near B_j. The lower bound will be $\min_j (1 + \|\nabla \phi_j\|_\infty^2)^{-1/2}$ where ϕ_j is the function which defines the boundary near B_j. Using a partition of unity $\sum_j \phi_j$ subordinate to the family of balls B_j which is 1 on $\partial\Omega$, we construct a vector field

$$\alpha(x) = \sum_{j=1}^{N} \phi_j(x)\alpha_j.$$

We have $\alpha(x) \cdot \nu(x) \ge \delta$ since each α_j satisfies this condition and each ϕ_j takes values in $[0, 1]$. Thus, the divergence theorem gives

$$\begin{aligned} \delta \int_{\partial\Omega} |u(x)|^2 \, d\sigma(x) &\le \int_{\partial\Omega} |u(x)|^2 \alpha(x) \cdot \nu(x) \, dx \\ &= \int_{\Omega} |u|^2 (\operatorname{div} \alpha) + 2 \operatorname{Re}(u(x)\alpha \cdot \nabla \bar{u}(x)) \, dx. \end{aligned}$$

Applying the Cauchy-Schwarz inequality proves the inequality of the Lemma. The constant depends on Ω through the vector field α and its derivatives. ∎

Proof of Proposition 10.14. The proposition follows from the lemma. That the map r can be extended from nice functions to all of $L^{2,1}(\Omega)$ depends on Lemma 10.10 which asserts that nice functions are dense in $L^{2,1}(\Omega)$. ∎

Exercise 10.18 *Suppose that Ω is a C^1 domain. Show that if $\phi \in C^1(\bar{\Omega})$ and $\phi(x) = 0$ on $\partial\Omega$, then $\phi(x)$ is in the Sobolev space $L^{2,1}_0(\Omega)$.*

Finally, we extend the definition of one of the Sobolev spaces of negative order to domains. We define $L^{2,-1}(\Omega)$ to be the dual of the space $L^{2,1}_0(\Omega)$. As in the case of $\mathbf{R}^n$, the following simple lemma gives examples of elements in this space.

Proposition 10.19 *Assume Ω is an open set of finite measure, and g and $f_1, \ldots, f_n$ are functions in $L^2(\Omega)$. Then*

$$\phi \to \lambda(\phi) = \int_\Omega g(x)\phi(x) + \sum_{j=1}^n f_j(x)\frac{\partial \phi(x)}{\partial x_j}\, dx$$

is in $L^{2,-1}(\Omega)$.

Proof. According to the Cauchy-Schwarz inequality, we have

$$\lambda(\phi) \leq \left(\int_\Omega |u(x)|^2 + |\nabla u(x)|^2\, dx\right)^{1/2} \left(\int_\Omega |g(x)|^2 + |\sum_{j=1}^n f_j(x)^2|\, dx\right)^{1/2}.$$

∎

10.2 The weak Dirichlet problem

In this section, we introduce elliptic operators. We let $A(x)$ be function defined on an open set Ω and we assume that this function takes values in $n \times n$-matrices with real entries. We assume that each entry is Lebesgue measurable and that A satisfies the symmetry condition

$$A^t = A \tag{10.20}$$

and ellipticity condition, for some $\lambda > 0$,

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda^{-1}|\xi|^2, \qquad \xi \in \mathbf{R}^n,\ x \in \Omega. \tag{10.21}$$

We say that u is a local weak solution of the equation $\operatorname{div} A(x)\nabla u = f$ for $f \in L^{2,-1}(\Omega)$ if u is in $L^{2,1}_{loc}(\Omega)$ and for all test functions $\phi \in \mathcal{D}(\Omega)$, we have

$$-\int_\Omega A(x)\nabla u(x) \cdot \nabla\phi(x)\,dx = f(\phi).$$

Since the derivatives of u are locally in L^2, we can extend to test functions ϕ which are in $L^{2,1}_0(\Omega)$ and which (have a representative) which vanishes outside a compact subset of Ω. However, let us resist the urge to introduce yet another space.

Statement of the Dirichlet problem. The weak formulation of the Dirichlet problem is the following. Let $g \in L^{2,1}(\Omega)$ and $f \in L^{2,-}1(\Omega)$, then we say that u is a solution of the Dirichlet problem if the following two conditions hold:

$$u \in L^{2,1}(\Omega) \tag{10.22}$$

$$u - g \in L^{2,1}_0(\Omega) \tag{10.23}$$

$$-\int_\Omega A(x)\nabla u(x)\nabla\phi(x)\,dx = f(\phi) \qquad \phi \in L^{2,1}_0(\Omega). \tag{10.24}$$

Note that both sides of the equation (10.24) are continuous in ϕ in the topology of $L^{2,1}_0(\Omega)$. Thus, we only need to require that this hold for ϕ in a dense subset of $L^{2,1}_0(\Omega)$.

A more traditional way of writing the Dirichlet problem is, given g and f find u which satisfies

$$\begin{cases} \operatorname{div} A\nabla u = f, & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \end{cases}$$

Our condition (10.24) is a restatement of the equation, $\operatorname{div} A\nabla u = f$. The condition (10.23) is a restatement of the boundary condition $u = f$. Finally, the condition (10.22) is needed to show that the solution is unique.

Theorem 10.25 *If Ω is an open set of finite measure and $g \in L^{2,1}(\Omega)$ and $f \in L^{2,-1}(\Omega)$, then there is exactly one weak solution to the Dirichlet problem, (10.22-10.24). There is a constant $C(\lambda, n, \Omega)$ so that the solution u satisfies*

$$\|u\|_{L^{2,1}(\Omega)} \leq C(\|g\|_{L^{2,1}(\Omega)} + \|f\|_{L^{2,-1}(\Omega)}).$$

Proof. *Existence.* If $u \in L^{2,1}_0(\Omega)$ and $n \geq 3$ then Hölder's inequality and then the Sobolev inequality of Theorem 8.22 imply

$$\int_\Omega |u(x)|^2\,dx \leq \left(\int_\Omega |u(x)|^{\frac{2n}{n-2}}\,dx\right)^{1-\frac{2}{n}} m(\Omega)^{2/n} \leq Cm(\Omega)^{2/n}\int_\Omega |\nabla u(x)|^2\,dx.$$

If $n = 2$, the same result holds, though we need to be a bit more persistent and use Hölder's inequality, the Soboleve inequality and Hölder again to obtain:

$$\begin{aligned}\int_\Omega |u(x)|^2\,dx \le \left(\int_\Omega |u(x)|^4\,dx\right)^{1/2} m(\Omega)^{1/2} &\le \left(\int_\Omega |\nabla u(x)|^{4/3}\,dx\right)^{3/2} m(\Omega)^{1/2} \\ &\le \int_\Omega |\nabla u(x)|^2\,dx\ m(\Omega).\end{aligned}$$

Note that in each case, the application of the Sobolev inequality on $\mathbf{R}^n$ is allowed because $L^{2,1}_0(\Omega)$ may be viewed as a subspace of $L^{2,1}(\Omega)$ by extending functions on Ω to be zero outside Ω. Thus we have

$$\|u\|_{L^2(\Omega)} \le Cm(\Omega)^{1/n}\|\nabla u\|_{L^2(\Omega)}. \tag{10.26}$$

Next, we observe that the ellipticity condition (10.21) implies that

$$\lambda \int_\Omega |\nabla u(x)|^2 \le \int_\Omega A(x)\nabla u(x)\nabla \bar{u}(x)\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2\,dx. \tag{10.27}$$

We claim the expression

$$\int_\Omega A(x)\nabla u(x)\nabla \bar{v}(x)\,dx \tag{10.28}$$

provides an inner product on $L^{2,1}_0(\Omega)$ which induces the same topology as the standard inner product on $L^{2,1}_0(\Omega) \subset L^{2,1}(\Omega)$ defined in (10.4). To see that the topologies are the same, it suffices to establish the inequalities

$$\int_\Omega |\nabla u(x)|^2 + |u(x)|^2\,dx \le \lambda^{-1}(1 + Cm(\Omega)^{2/n})\int_\Omega A(x)\nabla u(x)\nabla\bar{u}(x)\,dx$$

and that

$$\int_\Omega A(x)\nabla u(x)\nabla \bar{u}(x)\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2\,dx \le \lambda^{-1}\int_\Omega |\nabla u(x)|^2 + |u(x)|^2\,dx.$$

These both follow from the estimates (10.26) and (10.27). As a consequence, standard Hilbert space theory tells us that any continuous linear functional on $L^{2,1}_0(\Omega)$ can be represented using the inner product defined in (10.28). We apply this to the functional

$$\phi \to -\int_\Omega A\nabla g\nabla\phi\,dx - f(\phi)$$

and conclude that there exists $v \in L^{2,1}_0(\Omega)$ so that

$$\int_\Omega A(x)\nabla v(x)\nabla\phi(x)\,dx = -\int_\Omega A(x)\nabla g(x)\nabla\phi(x)\,dx - f(\phi), \qquad \phi \in L^2_{1,0}(\Omega). \quad (10.29)$$

Rearranging this expression, we can see that $u = g + v$ is a weak solution to the Dirichlet problem.

Uniqueness. If we have two solutions of the Dirichlet problem u_1 and u_2, then their difference $w = u_1 - u_2$ is a weak solution of the Dirichlet problem with $f = g = 0$. In particular, w is in $L^{2,1}_0(\Omega)$ and we can use $\bar{w}$ as a test function and conclude that

$$\int A(x)\nabla w(x)\cdot\nabla\bar{w}(x)\,dx = 0.$$

Thanks to the inequalities (10.26) and (10.27) we conclude that

$$\int_\Omega |w(x)|^2\,dx = 0.$$

Hence, $u_1 = u_2$.

Stability. Finally, we establish the estimate for the solution. We replace the test function ϕ in (10.29) by $\bar{v}$. Using the Cauchy-Schwarz inequality gives

$$\int_\Omega A\nabla v\cdot\nabla\bar{v}\,dx \le \lambda^{-1}\|v\|_{L^{2,1}_0(\Omega)}\|\nabla g\|_{L^{2,1}(\Omega)} + \|f\|_{L^{2,-1}(\Omega)}\|v\|_{L^{2,1}_0(\Omega)}.$$

If we use that the left-hand side of this inequality is equivalent with the norm in $L^{2,1}_0(\Omega)$, cancel the common factor, we obtain that

$$\|v\|_{L^{2,1}_0(\Omega)} \le C\|g\|_{L^{2,1}_0(\Omega)} + \|f\|_{L^{2,-1}(\Omega)}.$$

We have $u = v + g$ and the triangle inequality gives

$$\|u\|_{L^{2,1}(\Omega)} \le \|g\|_{L^{2,1}(\Omega)} + \|v\|_{L^{2,1}_0(\Omega)}$$

so combining the last two inequalities implies the estimate of the theorem. ∎

Exercise 10.30 *(Dirichlet's principle.) Let $g \in L^{2,1}(\Omega)$ and suppose that $f = 0$ in the weak formulation of the Dirichlet problem.*

a) Show that the expression

$$I(u) = \int_\Omega A(x)\nabla u(x)\cdot\nabla\bar{u}(x)\,dx$$

attains a minimum value on the set $g + L_0^{2,1}(\Omega) = \{g + v : v \in L_0^{2,1}(\Omega)\}$*. Hint: Use the foil method. This is a general fact in Hilbert space.*

b) If u *is a minimizer for* I*, then* u *is a weak solution of the Dirichlet problem,* $\operatorname{div} A\nabla u = 0$ *and* $u = g$ *on the boundary.*

c) Can you extend this approach to solve the general Dirichlet problem $\operatorname{div} A\nabla u = f$ *in* Ω *and* $u = g$ *on the boundary?*

Chapter 11

Inverse Problems: Boundary identifiability

11.1 The Dirichlet to Neumann map

In this section, we introduce the Dirichlet to Neumann map. Recall the space $L^{2,1/2}(\partial\Omega)$ which was introduced in Chapter 10. We let Ω be a bounded open set, A a matrix which satisfies the ellipticity condition and given f in $L^{2,1/2}(\partial\Omega)$, we let $u = u_f$ be the weak solution of the Dirichlet problem

$$\left\{ \begin{array}{ll} \operatorname{div} A\nabla u = 0, & \text{on } \Omega \\ u = f, & \text{on } \partial\Omega. \end{array} \right. \tag{11.1}$$

Given $u \in L^{2,1}(\Omega)$ we can define a continuous linear functional on $L^{2,1}(\Omega)$ by

$$\phi \to \int_\Omega A(x)\nabla u(x)\nabla\phi(x)\,dx.$$

If we recall the Green's identity (10.3), we see that if u and A are smooth, then

$$\int_{\partial\Omega} A(x)\nabla u(x)\cdot\nu(x)\phi(x)\,d\sigma(x) = \int_\Omega A(x)\nabla u(x)\nabla\phi(x) + \phi(x)\operatorname{div} A(x)\nabla u(x)\,dx.$$

Thus, if u solves the equation $\operatorname{div} A\nabla u = 0$, then it reasonable to define $A\nabla u\cdot\nu$ as a linear functional on $L^{2,1/2}(\partial\Omega)$ by

$$A\nabla u\cdot\nu(\phi) = \int_\Omega A(x)\nabla u(x)\cdot\nabla\phi(x)\,dx. \tag{11.2}$$

We will show that this map is defined on $L^{2,1/2}(\partial\Omega)$ The expression $A\nabla u \cdot \nu$ is called the *conormal derivative* of u at the boundary. Note that it is something of a miracle that we can make sense of this expression at the boundary. To appreciate that this is surprising, observe that we are not asserting that the full gradient of u is defined at the boundary, only the particular component $A\nabla u \cdot \nu$. The gradient of u may only be in $L^{2,(}(\Omega)$ and thus there is no reason to expect that any expression involving ∇u could make sense on the boundary, a set of measure zero.

A potential problem is that this definition may depend on the representative of ϕ which is used to define the right-hand side of (11.2). Fortunately, this is not the case.

Lemma 11.3 *If $u \in L^{2,1}(\Omega)$ and u is a weak solution of* $\operatorname{div} A\nabla u = 0$*, then the value of $A\nabla u \cdot \nu(\phi)$ is independent of the extension of ϕ from $\partial\Omega$ to Ω.*

The linear functional defined in (11.2) is a continuous linear functional on $L^{2,1/2}(\partial\Omega)$.

Proof. To establish that $A\nabla u \cdot \nu$ is well defined, we will use that u is a solution of $\operatorname{div} A\nabla u = 0$. We choose ϕ_1, ϕ_2 in $L^{2,1}(\Omega)$ and suppose $\phi_1 - \phi_2 \in L^{2,1}_0(\Omega)$. According to the definition of weak solution,

$$\int_\Omega A(x)\nabla u(x) \cdot \nabla(\phi_1(x) - \phi_2(x))\, dx = 0.$$

To establish the continuity, we need to choose a representative of ϕ which is close to the infinum in the definition of the $L^{2,1/2}$-norm (see (10.13)). Thus we need $\|\phi\|_{L^{2,1}(\Omega)} \leq 2\|r\phi\|_{L^{2,1/2}(\partial\Omega)}$. Here, $r\phi$ denotes the restriction of ϕ to the boundary. With this choice of ϕ and Cauchy-Schwarz we have

$$|A\nabla u \cdot \nu(\phi)| \leq C\|\nabla u\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}.$$

This inequality implies the continuity. ∎

We will define $L^{2,-1/2}(\partial\Omega)$ as the dual of the space $L^{2,1/2}(\partial\Omega)$. Now, we are ready to define the *Dirichlet to Neumann map.* This is a map

$$\Lambda_A : L^{2,1/2}(\Omega) \to L^{2,-1/2}(\partial\Omega)$$

defined by

$$\Lambda_A f = A\nabla u \cdot \nu$$

where u is the solution of the Dirichlet problem with boundary data f.

The traditional goal in pde is to consider the direct problem. For example, given the coefficient matrix A, show that we can solve the Dirichlet problem. If we were more persistent, we could establish additional properties of the solution. For example, we could show that the map $A \to \Lambda_A$ is continuous, on the set of strictly positive definite matrix valued functions on Ω.

However, that would be the easy way out. The more interesting and difficult problem is the inverse problem. Given the map Λ_A, can we recover the coefficient matrix, A. That is given some information about the solutions to a pde, can we recover the equation. The answer to the problem, as stated, is no, of course not.

Exercise 11.4 *Let Ω be a bounded domain and let $F : \Omega \to \Omega$ be a $C^1(\bar{\Omega})$ diffeomorphism that fixes a neighborhood of the boundary. Show that if A gives an elliptic operator $\operatorname{div} A\nabla$ on Ω, then there is an operator $\operatorname{div} B\nabla$ so that*

$$\operatorname{div} A\nabla u = 0 \iff \operatorname{div} B\nabla u \circ F = 0.$$

As a consequence, it is clear that the maps $\Lambda_A = \Lambda_B$. Hint: See Lemma 11.10 below for the answer.

Exercise 11.5 *Show that the only obstruction to uniqueness is the change of variables described in the previous problem.*

Remark: This has been solved in two dimensions, by John Sylvester [35]. In three dimensions and above, this problem is open.

Exercise 11.6 *Prove that the map $A \to \Lambda_A$ is continuous on the set of strictly positive definite and bounded matrix-valued functions. That is show that*

$$\|\Lambda_A - \Lambda_B\|_{\mathcal{L}(L^{2,1/2},L^{2,-1/2})} \leq C_\lambda \|A - B\|_\infty.$$

Here, $\|\cdot\|_{\mathcal{L}(L^{2,1/2},L^{2,-1/2})}$ denotes the norm on linear operators from $L^{2,1/2}$ to $L^{2,-1/2}$.

a) As a first step, show that if we let u_A and u_B satisfy $\operatorname{div} A\nabla u_A = \operatorname{div} B\nabla u_B = 0$ in an open set Ω and $u_A = u_B = f$ on $\partial\Omega$, then we have

$$\int_\Omega |\nabla u_A - \nabla u_B|^2 \, dx \leq C\|f\|_{L^{2,1/2}(\partial\Omega)} \|A - B\|_\infty^2.$$

Hint: We have $\operatorname{div} B\nabla u_A = \operatorname{div}(B - A)\nabla u_A$ since u_A is a solution.

b) Conclude the estimate above on the Dirichlet to Neumann maps.

However, there is a restricted version of the inverse problem which can be solved. In the remainder of these notes, we will concentrate on elliptic operators when the matrix A is of the form $A(x) = \gamma(x)I$ where I is the $n \times n$ identity matrix and $\gamma(x)$ is a scalar function which satisfies

$$\lambda \leq \gamma(x) \leq \lambda^{-1} \tag{11.7}$$

for some constant $\lambda > 0$. We change notation a bit and let Λ_γ be the Dirichlet to Neumann map for the operator $\operatorname{div} \gamma \nabla$. Then the *inverse conductivity problem* can be formulated as the following question:

$$\text{Is the map } \gamma \to \Lambda_\gamma \text{ injective?}$$

We will answer this question with a yes, if the dimension $n \geq 3$ and we have some reasonable smoothness assumptions on the domain and γ. This is a theorem of J. Sylvester and G. Uhlmann [36]. The following year, closely related work was done by Henkin and R. Novikov [17, 23]. One can also ask for a more or less explicit construcion of the inverse map. A construction is given in in the work of Novikov and the work of A. Nachman [22] for three dimensions and [21] in two dimensions. This last paper also gives the first proof of injectivity in two dimensions. My favorite contribution to this story is in [9]. But this is not the the place for a complete history.

We take a moment to explain the appearance of the word conductivity in the above. For this discussion, we will assume that function u and γ are smooth. The problem we are considering is a mathematical model for the problem of determining the conductivity γ by making measurements of current and voltage at the boundary. To try and explain this, we suppose that u represents the voltage potential in Ω and then ∇u is the electric field. The electric field is what makes electrons flow and thus we assume that the current is proportional to the electric field, $J = \gamma \nabla u$ where the conductivity γ is a constant of proportionality. Since we assume that charge is conserved, for each subregion $B \subset \Omega$, the net flow of electrons or current through B must be zero. Thus,

$$0 = \int_{\partial B} \gamma \nabla u(x) \cdot \nu(x)\, d\sigma(x).$$

The divergence theorem gives that

$$0 = \int_{\partial B} \gamma(x) \nabla u(x) \cdot \nu(x)\, d\sigma(x) = \int_B \operatorname{div} \gamma(x) \nabla u(x)\, dx.$$

Finally, since the integral on the right vanishes, say, for each ball $B \subset \Omega$, we can conclude that $\operatorname{div} \gamma \nabla u = 0$ in Ω.

11.2 Identifiability

Our solution of the inverse conductivity problem has two steps. The first is to show that the Dirichlet to Neumann map determines γ on the boundary. The second step is to use the knowledge of γ on the boundary to relate the inverse conductivity problem to a problem in all of $\mathbf{R}^n$ which turns out to be a type of scattering problem. We will use the results of Chapter 9 to study this problem in $\mathbf{R}^n$.

Theorem 11.8 *Suppose that $\partial\Omega$ is C^1. If γ is in $C^0(\bar{\Omega})$ and satisfies (11.7), then for each $x \in \partial\Omega$, there exists a sequence of functions u_N so that*

$$\gamma(x) = \lim_{N\to\infty} \Lambda_\gamma u_N(\bar{u}_N).$$

Theorem 11.9 *Suppose Ω and γ are as in the previous theorem and also $\partial\Omega$ is C^2 and γ is in $C^1(\bar{\Omega})$. If e is a constant vector and u_N as in the previous theorem, then we have*

$$\nabla\gamma(x)\cdot e = \lim_{N\to\infty} \int_{\partial\Omega} \left(\gamma(x)|\nabla u_N(x)|^2 e\cdot\nu(x) - 2\operatorname{Re}\gamma(x)\frac{\partial u_N}{\partial\nu}(x)e\cdot\nabla\bar{u}(x)\right)\, d\sigma.$$

The construction of the solutions u_N proceeds in two steps. The first step is to write down an explicit function which is an approximate solution and show that the conclusion of our Theorem holds for this function. The second step is to show that we really do have an approximate solution. This is not deep, but requires a certain amount of persistence. I say that the result is not deep because it relies only on estimates which are a byproduct of our existence theory in Theorem 10.25.

In the construction of the solution, it will be convenient to change coordinates so that in the new coordinates, the boundary is flat. The following lemma keeps track of how the operator $\operatorname{div}\gamma\nabla$ transforms under a change of variables.

Lemma 11.10 *Let A be an elliptic matrix and $F:\Omega'\to\Omega$ be a $C^1(\bar{\Omega}')$-diffeomorphism. We have that $\operatorname{div} A\nabla u = 0$ if and only if $\operatorname{div} B\nabla u\circ F$ where*

$$B(y) = |\det DF(y)| DF^{-1}(F(y))^t A(F(y)) DF^{-1}(F(y)).$$

Proof. The proof of this lemma indicates one of the advantages of the weak formulation of the equation. Since the weak formulation only involves one derivative, we only need to use the chain rule once.

We use the chain rule to compute

$$\nabla u(x) = \nabla(u(F(F^{-1}(x))) = DF^{-1}(x)\nabla(u\circ F)(F^{-1}(x)).$$

This is valid for Sobolev functions also by approximation (see Lemma 10.11). We insert this expression for the gradient and make a change of variables $x = F(y)$ to obtain

$$\begin{aligned}
&\int_\Omega A(x)\nabla u(x)\cdot\nabla\phi(x)\,dx \\
&= \int_{\Omega'} A(F(y))DF^{-1}(F(y))\nabla(u\circ F(y))\cdot DF^{-1}(F(y))\nabla(\phi\circ F(y))|\det DF(y)|\,dy \\
&= \int_{\Omega'} |\det DF(y)|DF^{-1}(F(y))^t A(F(y))DF^{-1}(F(y))\nabla(u\circ F(y))\cdot\nabla(\phi\circ F(y))\,dy.
\end{aligned}$$

This last integral is the weak formulation of the equation $\operatorname{div} B\nabla u = 0$ with the test function $\phi\circ F$. To finish the proof, one must convince oneself that the map $\phi\to\phi\circ F$ is an isomorphism[1] from $L^{2,1}_0(\Omega)$ to $L^{2,1}_0(\Omega')$. ∎

Exercise 11.11 *Figure out how to index the matrix DF^{-1} so that in the application of the chain rule in the previous Lemma, the product $DF^{-1}\nabla(u\circ F)$ is matrix multiplication. Assume that the gradient is a column vector.*

Solution The chain rule reads

$$\frac{\partial}{\partial x_i} u\circ G = \frac{\partial G_j}{\partial x_i}\frac{\partial u}{\partial x_j}\circ G.$$

Thus, we want

$$(DG)_{ij} = \frac{\partial G_j}{\partial x_i}.$$

In the rest of this chapter, we fix a point x on the boundary and choose coordinates so that x is the origin. Thus, we suppose that we are trying to find the value of γ and $\nabla\gamma$ at 0. We assume that $\partial\Omega$ is C^1 near 0 and thus we have a ball $B_r(0)$ so that $B_{2r}(0)\cap\partial\Omega = \{(x', x_n) : x_n = \phi(x')\}\cap B_{2r}(0)$. We let $x = F(y', y_n) = (y', \phi(y') + y_n)$. Note that we assume that the function ϕ is defined in all of $\mathbf{R}^{n-1}$ and thus, the map F is invertible on all of $\mathbf{R}^n$. In the coordinates, (y', y_n), the operator $\operatorname{div}\gamma\nabla$ takes the form

$$\operatorname{div} A\nabla u = 0$$

with $A(y) = \gamma(y)B(y)$. (Strictly speaking, this is $\gamma(F(y))$. However, to simplify the notation, we will use $\gamma(z)$ to represent the value of γ at the point corresponding to z

[1] An *isomorphism* for Banach (or Hilbert spaces) is an invertible linear map with continuous inverse. A map which also preserves the norm is called an *isometry.*

in the current coordinate system. This is a fairly common convention. To carry it out precisely would require yet another chapter that we don't have time for...) The matrix B depends on ϕ and, by the above lemma, takes the form

$$B(y) = \begin{pmatrix} 1_{n-1} & -\nabla\phi(y') \\ -\nabla\phi(y')^t & 1 + |\nabla\phi(y')|^2 \end{pmatrix}.$$

Apparently, we are writing the gradient as a column vector. The domain Ω' has 0 on the boundary and near 0, $\partial\Omega'$ lies in the hyperplane $y_n = 0$ and Ω' lies in the regions $y_n > 0$. We introduce a real-valued cutoff function $\eta(y) = f(y')g(y_n)$ where f is supported in $|y'| < 2$ and is normalized so that

$$\int_{\mathbf{R}^{n-1}} f(y')^2\, dy' = 1 \tag{11.12}$$

and so that $g(y_n) = 1$ if $|y_n| < 1$ and $g(y_n) = 0$ if $|y_n| > 2$. Our next step is to set $\eta_N(x) = N^{(n-1)/4}\eta(N^{1/2}y)$. We choose a vector $\alpha \in \mathbf{R}^n$ and which satisfies

$$B(0)\alpha \cdot e_n = 0 \tag{11.13}$$

$$B(0)\alpha \cdot \alpha = B(0)e_n \cdot e_n. \tag{11.14}$$

We define E_N by

$$E_N(y) = N^{-1/2} \exp(-N(y_n + i\alpha \cdot y))$$

and then we put

$$v_N(y) = \eta_N(y)E_N(y). \tag{11.15}$$

The function v_N is our approximate solution. The main facts that we need to prove about v_N are Lemma 11.18 and Lemma 11.25 below. Lemma 11.25 asserts that v_N is an appoximate solution of an elliptic equation. To visualize why this might be true, observe that E_N is a solution of the equation with constant coefficients $B(0)$. The cutoff function oscillates less rapidly than E_N (consider the relative size of the gradients) and thus it introduces an error that is negligible for N large and allows us to disregard the fact that E_N is not a solution away from the origin.

Our proof will require yet more lemmas. The function v_N is concentrated near the boundary. In the course of making estimates, we will need to consider integrals pairing v_N and its derivatives against functions which are in $L^{2,1}_0(\Omega)$. To make optimal estimates, we will want to exploit the fact that functions in $L^{2,1}_0(\Omega)$ are small near the boundary. The next estimate, a version of *Hardy's inequality* makes this precise. If we have not already made this definition, then we define

$$\delta(x) = \inf_{y\in\partial\Omega} |x - y|.$$

The function δ gives the distance from x to the boundary of Ω.

Lemma 11.16 *(Hardy's inequality) a) Let f be a C^1 function on the real line and suppose that $f(0) = 0$, then for $1 < p \le \infty$,*

$$\int_0^\infty |\frac{f(t)}{t}|^p \, dt \le p' \int_0^\infty |f'(t)|^p \, dt.$$

b) If f is in $L_0^{2,1}(\Omega)$, then

$$\int_\Omega \left|\frac{f(x)}{\delta(x)}\right|^2 \, dx \le C \int_\Omega |\nabla f(x)|^2 + |f(x)|^2 \, dx.$$

Proof. *a)* We prove the one-dimensional result with $p < \infty$ first. We use the fundamental theorem of calculus to write

$$f(t) = -\int_0^t f'(s) \, ds$$

Now, we confuse the issue by rewriting this as

$$\begin{aligned} -t^{\frac{1}{p}-1} f(t) &= \int (t/s)^{-1/p'} \chi_{(1,\infty)}(t/s) s^{1/p} f(s) \, \frac{ds}{s} \\ &= \int K(t/s) s^{1/p} f'(s) \, \frac{ds}{s} \end{aligned} \tag{11.17}$$

where $K(u) = u^{-1/p'} \chi_{(1,\infty)}(u)$. A computation shows that

$$\int_0^\infty K(t/s) \frac{ds}{s} = \int_0^\infty K(t/s) \, \frac{dt}{t} = p'$$

which will be finite if $p > 1$. Thus, by exercise 4.5 we have that $g \to \int K(t/s) g(s) \, ds/s$ maps $L^p(ds/s)$ into itself. Using this in (11.17) gives

$$\left(\int_0^\infty \left|\frac{f(t)}{t}\right|^p \, dt\right)^{1/p} \le p' \left(\int_0^\infty |f'(t)|^p \, dt\right)^{1/p}.$$

Which is what we wanted to prove. The remaining case $p = \infty$ where the L^p norms must be replaced by L^∞ norms is easy and thus omitted.

b) Since $\mathcal{D}(\Omega)$ is dense in $L_0^{2,1}(\Omega)$, it suffices to consider functions in $\mathcal{D}(\Omega)$. By a partition of unity, as in Lemma 10.8 we can further reduce to a function f which is compactly supported $B_r(x) \cap \Omega$, for some ball centered at x on the boundary, or to a function f which is supported at a fixed distance away from the boundary. In the first case have that $\partial\Omega$ is given by the graph $\{(y', y_n) : y_n = \phi(y')\}$ near x. Applying the one-dimensional result in the y_n variable and then integrating in the remaining variables, we may conclude that

$$\int_{\Omega\cap B_r(x)} \frac{|f(y)|^2}{(y_n - \phi(y'))^2}\, dy \leq 4 \int_{\Omega\cap B_r(x)} |\frac{\partial u}{\partial y_n}(y)|^2\, dy.$$

This is the desired inequality once we convince ourselves that $(y_n - \phi(y'))/\delta(y)$ is bounded above and below in $B_r(x) \cap \Omega$.

The second case where f is supported strictly away from the boundary is an easy consequence of the Sobolev inequality, Theorem 8.22, because $1/\delta(x)$ is bounded above on each compact subset of Ω. ∎

The following Lemma will be useful in obtaining the properties of the approximate solutions and may serve to explain some of the peculiar normalizations in the definition.

Lemma 11.18 *Let v_N, E_N and η_N be as defined in (11.15). Let β be continuous at 0 then*

$$\lim_{N\to\infty} N \int_{\Omega'} \beta(y)|\eta_N(y)|^2 e^{-2Ny_n}\, dy = \beta(0)/2. \tag{11.19}$$

If $k > -1$ and $\tilde{\eta} \in \mathcal{D}(\mathbf{R}^n)$, then for N sufficiently large there is a constant C so that

$$|\int_{\Omega'} \delta(y)^k \tilde{\eta}(N^{1/2}y) e^{-2Ny_n}\, dy| \leq CN^{\frac{1-n}{2}-1-k}. \tag{11.20}$$

Proof. To prove the first statement, we observe that by the definition and the normalization of the cutoff function, f, in (11.12) we have that

$$\begin{aligned}\int_{\Omega'} \eta_N(y)^2 e^{-2Ny_n}\, dy &= N^{\frac{n-1}{2}} \int_{\{y:y_n>0\}} f(N^{1/2}y')^2 e^{-2Ny_n}\, dy \\ &\quad + N^{\frac{n-1}{2}} \int_{\{y:y_n>0\}} (g(N^{1/2}y_n)^2 - 1) f(N^{1/2}y')^2 e^{-2Ny_n}\, dy.\end{aligned}$$

The first integral is $1/(2)$ and the second is bounded by a multiple of $(2N)^{-1}e^{-2N^{1/2}}$. The estimate of the second depends on our assumption that $g(t) = 1$ for $t < 1$. Thus, we have that

$$\lim_{N\to\infty} N \int_{\Omega'} \eta_N(y)^2 e^{-2Ny_n} dy = 1/2.$$

Using this to express the $\frac{1}{2}$ as a limit gives

$$\begin{aligned}|\frac{1}{2}\beta(0) - \lim_{N\to\infty} N\int_{\Omega'} \beta(y)\eta_N(y)^2 e^{-2Ny_n} dy| &\leq \lim_{N\to\infty} N\int_{\Omega'} |\beta(0)-\beta(y)| \\ &\quad \times \eta_N(y)^2 e^{-2Ny_n} dy \\ &\leq \lim_{N\to\infty} \sup_{\{y:|y|<2^{1/2}N^{-1/2}\}} \frac{1}{2}|\beta(0)-\beta(y)| \\ &\quad \times N\int_{\Omega'} \eta_N(y)^2 e^{-2Ny_n} dy.\end{aligned}$$

Now the continuity of β implies that this last limit is 0.

The inequalities in the second statement follow easily, by observing that for N sufficiently large, we have $\delta(y) = y_n$ on the support of $\tilde{\eta}(N^{1/2}y)$. If $\operatorname{supp}\tilde{\eta} \subset B_R(0)$, then we can estimate our integral by

$$\begin{aligned}\left|\int_{\Omega'} \delta(y)^k \tilde{\eta}(N^{1/2}y)e^{-2Ny_n}\, dy\right| &\leq \|\tilde{\eta}\|_\infty \int_{\{y':|y'|<N^{-1/2}R\}} \int_0^\infty y_n^k e^{-2Ny_n}\, dy' dy_n \\ &\leq CN^{\frac{1-n}{2}-1-k}.\end{aligned}$$

∎

We can now evaluate a limit involving our approximate solution.

Lemma 11.21 *With Ω' and v_N as above, suppose β is a bounded function on Ω' which is continuous at 0, then*

$$\lim_{N\to\infty} \int_{\Omega'} \beta(y)B(y)\nabla v_N(y) \cdot \nabla \bar{v}_N(y)\, dy = \beta(0)B(0)e_n \cdot e_n.$$

Proof. Using the product rule, expanding the square and that η_N is real valued gives

$$\begin{aligned}\int_{\Omega'} \beta(y)B(y)\nabla v_N(y)\nabla \bar{v}_N(y)\, dy &= N\int \beta(y)(B(y)\alpha\cdot\alpha + B(y)e_n\cdot e_n)\eta_N(y)^2 e^{-2Ny_n}\, dy \\ &\quad -2\int \beta(y)(B(y)\nabla\eta_N(y)\cdot e_n)\eta_N(y)e^{-2Ny_n}\, dy \\ &\quad +N^{-1}\int_{\Omega'} \beta(y)B(y)\nabla\eta_N(y)\cdot\nabla\eta_N(y)e^{-2Ny_n}\, dy \\ &= I + II + III.\end{aligned}$$

By (11.19) of our Lemma 11.18, we have that

$$\lim_{N\to\infty} I = \beta(0)(B(0)e_n \cdot e_n). \tag{11.22}$$

where we have used (11.14) to replace $B(0)\alpha\cdot\alpha$ by $B(0)e_n\cdot e_n$. The integral II can be bounded above by

$$II \le 2N^{\frac{n}{2}}\|\beta B\|_\infty \int_{\Omega'} |(\nabla\eta_1)(N^{1/2}y)\eta_1(N^{1/2}y)|e^{-2Ny_n}\,dy \le CN^{-1/2}. \tag{11.23}$$

Here, we are using the second part of Lemma 11.18, (11.20). The observant reader will note that we have taken the norm of the matrix B in the above estimate. The estimate above holds if matrices are normed with the operator norm–though since we do not care about the exact value of the constant, it does not matter so much how matrices are normed.

Finally, the estimate for III also follows from (11.20) in Lemma 11.18 as follows:

$$III \le N^{\frac{n-1}{2}}\|\beta B\|_\infty \int_{\Omega'} |(\nabla\eta_1)(N^{1/2}y)|^2 e^{-2Ny_n}\,dy \le CN^{-1}. \tag{11.24}$$

The conclusion of the Lemma follows from (11.22–11.24). ∎

Now, we can make precise our assertion that v_N is an approximate solution of the equation $\operatorname{div} A\nabla v = 0$.

Lemma 11.25 *With v_N and Ω' as above,*

$$\lim_{N\to\infty} \|\operatorname{div} A\nabla v_N\|_{L^{2,-1}(\Omega')} = 0.$$

Proof. We compute and use that $\operatorname{div} A(0)\nabla E_N = 0$ to obtain

$$\begin{aligned}
\operatorname{div} A(y)\nabla v_N(y) &= \operatorname{div}(A(y)-A(0))\nabla v_N(y) + \operatorname{div} A(0)\nabla v_N(y)\\
&= \operatorname{div}(A(y)-A(0))\nabla v_N(y)\\
&\qquad +2A(0)\nabla\eta_N(y)\nabla E_N(y) + E_N \operatorname{div} A(0)\nabla\eta_N(y)\\
&= I + II + III.
\end{aligned}$$

In the term I, the divergence must be interpreted as a weak derivative. To estimate the norm in $L^{2,-1}(\Omega)$, we must pair each of I through III with a test function ψ. With I, we use the definition of weak derivative and recall that η_N is supported in a small ball to obtain

$$\begin{aligned}|I(\psi)| &= \left|\int (A(y)-A(0))\nabla v_N(y)\cdot\nabla\psi(y)\,dy\right| \\ &\leq \sup_{|y|<2^{3/2}N^{-1/2}} |A(y)-A(0)|\|\nabla v_N\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}.\end{aligned}$$

This last expression goes to zero with N because A is continuous at 0 and according to (11.21) the $L^2(\Omega)$ of the gradient of v_N is bounded as $N\to 0$.

To make estimates for II, we multiply and divide by $\delta(y)$, use the Cauchy-Schwarz inequality, the Hardy inequality, Lemma 11.16, and then (11.20)

$$\begin{aligned}|II(\psi)| &= |\int_{\Omega'} 2A(0)\nabla\eta_N(y)\nabla E_N(y)\psi(y)\,dy| \\ &\leq \left(\int_{\Omega'}\left|\frac{\psi(y)}{\delta(y)}\right|^2 dy\right)^{1/2}\left(N^{\frac{n+3}{2}}\int_{\Omega'}\delta(y)^2|(\nabla\eta_1)(N^{1/2}y)|^2 e^{-2Ny_n}\,dy\right)^{1/2} \\ &\leq CN^{-1/2}\|\psi\|_{L^{2,1}_0(\Omega')}.\end{aligned}$$

Finally, we make estimates for the third term

$$\begin{aligned}|III(\psi)| &= \left|\int E_N(y)\,\mathrm{div}\,A(0)\nabla\eta_N(y)\,dy\right| \\ &\leq \left(\int_{\Omega'}\left|\frac{\psi(y)}{\delta(y)}\right|^2 dy\right)^{1/2}\left(N^{\frac{n+1}{2}}\int_{\Omega'}\delta(y)^2|(\,\mathrm{div}\,A(0)\nabla\eta_1)(N^{1/2}y)|^2 e^{-2Ny_n}\,dy\right)^{1/2} \\ &\leq C\|\psi\|_{L^{2,1}_0(\Omega')}N^{-1}.\end{aligned}$$

∎

Now, it is easy to patch up v_N to make it a solution, rather than an approximate solution.

Lemma 11.26 *With Ω' and B as above, we can find a family of solutions, w_N, of $\mathrm{div}\,A\nabla w_N = 0$ with $w_N - v_N \in L^{2,1}_0(\Omega')$ so that*

$$\lim_{N\to\infty}\int_{\Omega'}\beta(y)B(y)\nabla w_N(y)\cdot\nabla\bar{w}_N(y)\,dy = \beta(0)B(0)e_n\cdot e_n.$$

Proof. According to Theorem 10.25 we can solve the Dirichlet problem

$$\begin{cases} \operatorname{div} A\nabla\tilde{v}_N = -\operatorname{div} A\nabla v_N, & \text{in } \Omega' \\ \tilde{v}_N = 0, & \text{on } \partial\Omega \end{cases}$$

The solution $\tilde{v}_N$ will satisfy

$$\lim_{N\to\infty} \|\nabla\tilde{v}_N\|_{L^2(\Omega')} \leq \lim_{N\to\infty} C\|\operatorname{div} A\nabla v_n\|_{L^{2,-1}(\Omega')} = 0 \tag{11.27}$$

by the estimates from the existence theorem, Theorem 10.25 and the estimate of Lemma 11.25.

If we set $w_N = v_N + \tilde{v}_N$, then we have a solution with the correct boundary values and by (11.27) and Lemma 11.21

$$\begin{aligned} \lim_{N\to\infty} \int_{\Omega'} \beta(y)B(y)\nabla w_N(y)\cdot\nabla\bar{w}_N(y)\,dy &= \lim_{N\to\infty}\int_{\Omega'} A\nabla v_N(y)\cdot\nabla\bar{v}_N(y)\,dy \\ &= \beta(0)B(0)e_n\cdot e_n. \end{aligned}$$

∎

We will need another result from partial differential equations–this one will not be proven in this course. This Lemma asserts that solutions of elliptic equations are as smooth as one might expect.

Lemma 11.28 *If A is matrix with $C^1(\bar{\Omega})$ entries and Ω is a domain with C^2-boundary, then the solution of the Dirichlet problem,*

$$\begin{cases} \operatorname{div} A\nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

will satisfy

$$\|u\|_{L^{2,2}(\Omega)} \leq C\|f\|_{L^{2,2}(\Omega)}.$$

As mentioned above, this will not be proven. To obtain an idea of why it might be true. Let u be a solution as in the theorem. This, we can differentiate and obtain that $v = \partial u/\partial x_j$ satisfies an equation of the form $\operatorname{div}\gamma\nabla v = \operatorname{div}(\partial\gamma/\partial x_j)\nabla u$. The right-hand side is in $L^{2,-1}$ and hence it is reasonable to expect that v satisfies the energy estimates of Theorem 10.25. This argument cannot be right because it does not explain how the boundary data enters into the estimate. To see the full story, take MA633.

Finally, we can give the proofs of our main theorems.

Proof of Theorem 11.8 and Theorem 11.9. We let $F : \Omega' \to \Omega$ be the diffeomorphism used above and let $u_N = w_N \circ F^{-1}/(1+|\nabla\phi(0)|^2)$. According to the change of variables lemma, u_N will be a solution of the original equation, $\operatorname{div}\gamma\nabla u_N = 0$ in Ω. Also, the Dirichlet integral is preserved:

$$\int_\Omega \beta(x)|\nabla u_N(x)|^2\,dx = \frac{1}{1+|\nabla\phi(0)|^2}\int_{\Omega'} \beta(y)B(y)\nabla w_N(y)\cdot\nabla\bar{w}_N(y)\,dy.$$

Thus, the recovery of γ at the boundary follows from the result in Ω' of Lemma 11.26 and we have

$$\gamma(0) = \lim_{N\to\infty}\int_\Omega \gamma(x)|\nabla u_N(x)|^2\,dx = \lim_{N\to\infty}\Lambda_\gamma(u_N)(\bar{u}_N).$$

For the proof of the second theorem, we use the same family of solutions and the Rellich identity [25]:

$$\int_{\partial\Omega}\gamma(x)e\cdot\nu(x)|\nabla u_N(x)|^2 - 2\operatorname{Re}\gamma(x)\frac{\partial u}{\partial\nu}(x)\,e\cdot\nabla\bar{u}(x)\,dx = \int_\Omega e\cdot\nabla\gamma(x)|\nabla u_N(x)|^2\,dx.$$

This is proven by an application of the divergence theorem. The smoothness result in Lemma 11.28 is needed to justify the application of the divergence theorem: we need to know that u_N has two derivatives to carry this out. The full gradient of u_N is determined by the boundary values of u_N and the Dirichlet to Neumann map.

By Lemma 11.21, if $\gamma \in C^1(\bar{\Omega})$, we can take the limit of the right-hand side and obtain that

$$\frac{\partial\gamma}{\partial x_j}(0) = \lim_{N\to\infty}\int_\Omega \frac{\partial\gamma}{\partial x_j}(x)|\nabla u_N(x)|^2\,dx.$$

∎

Corollary 11.29 *If we have a C^2 domain and for two $C^1(\bar{\Omega})$ functions, $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ on the boundary and $\nabla\gamma_2 = \nabla\gamma_1$ on the boundary.*

Proof. The boundary values of the function u_N are independent of γ_j. The expression $\Lambda_\gamma u_N(\bar{u}_N)$ in Theorem 11.8 clearly depends only on u_N and the map Λ_γ. The left-hand side Theorem 11.9 depends only on γ and ∇u_N. Since ∇u_N which can be computed from u_N and the normal derivative of u_N. Hence, we can use Theorem 11.9 to determine $\nabla\gamma$ from the Dirichlet to Neumann map. ∎

Exercise 11.30 *If γ and $\partial\Omega$ are regular enough, can we determine the second order derivatives of γ from the Dirichlet to Neumann map?*

It is known that all derivatives of u are determined by the Dirichlet to Neumann map. I do not know if there is a proof in the style of Theorems 11.8 and 11.9 which tell how to compute second derivatives of γ by looking at some particular expression on the boundary.

Exercise 11.31 *If one examines the above proof, one will observe that there is more to be done. We made an arbitrary choice for the vector α and used α in the determination of one function, γ. We can use this observation to study boundary values of conductivities that are more general than γI.*

See the work of Alessandrini and Gaburro [2]. and Kang and Yun [19].

11.3 Notes

A uniqueness result for the boundary values of a conductivity was given by Kohn and Vogelius [20]. They show that if two Dirichlet to Neumann maps are equal, then all derivatives of the Dirichlet to Neumann map agree at the boundary. Sylvester and Uhlmann [37] give a constructive result. That is they show how to recover the conductivity from the Dirichlet to Neumann map, at least if the domain sufficiently smooth. Alessandrini [1] gives a proof that requires less regularity. The argument presented in this chapter is taken from the work of the author [6].

Chapter 12

Spaces adapted to the operator $\Delta + 2\zeta \cdot \nabla$

In this section, we show that we can recover the conductivity from the Dirichlet to Neumann map when the conductivity is continuously differentiable. The result and the method we give for recovery are from a recent manuscript of Haberman and Tataru, [15]. Haberman and Tataru build on the fundamental work of Sylvester and Uhlmann [36] who observed that a crucial step in solving the inverse problem is constructing solutions of a Schrödinger equation which are asymptotic to complex exponentials at infinity. Thus we consider harmonic exponentials of the form $\exp(x \cdot \zeta)$ with $\zeta \in \mathcal{V} = \{\zeta \in \mathbf{C}^n : \zeta \cdot \zeta = 0\}$. If we conjugate the Laplacian by $\exp(x \cdot \zeta)$, we obtain the operator

$$\Delta + 2\zeta \cdot \nabla.$$

The first innovation of Haberman and Tataru is to consider spaces which are adapted to the operator in the style of Bourgain's $X^{s,b}$ spaces.

12.1 Spaces adapted to $\Delta + 2\zeta \cdot \nabla$

Thus, we let $p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi$ be the symbol of the operator $\Delta + 2\zeta \cdot \nabla$. We define the space $\dot{X}^b_\zeta$ to be closure of the Schwartz space in the norm

$$\|f\|^2_{\dot{X}^b_\zeta} = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 |p_\zeta(\xi)|^{2b}\, d\xi$$

We will also consider the inhomogeneous version of this space, X^b_ζ which is normed by

$$\|f\|^2_{X^b_\zeta} = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 (|\zeta| + |p_\zeta(\xi)|)^{2b}\, d\xi$$

We will restrict our attention to $b = \pm 1/2$. It is clear that the operator $\Delta + 2\zeta \cdot \nabla$: $\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta$ is an isomorphism. Thus, the interesting part of studying these spaces will the study of operator $u \to qu$ on these spaces.

Our first result gives local regularity of the functions in the space $\dot{X}^{1/2}_\zeta$. We will need the following simple result on maps between weighted L^2 spaces. If w is a non-negative, Borel measurable function (or weight), we let $L^2(w)$ denote the collection of Borel measurable functions for which the norm

$$\|f\|^2_{L^2(w)} = \int_{\mathbf{R}^n} |f(\xi)|^2 w(\xi)\, d\xi$$

is finite.

Lemma 12.1 *Let ϕ be in $L^1(\mathbf{R}^n)$ and let v and w be weights on $\mathbf{R}^n$. Define Tf by*

$$Tf(\xi) = \int_{\mathbf{R}^n} \phi(\xi - \eta) f(\eta)\, d\eta.$$

We have

$$\|Tf\|_{L^2(w)} \leq A \|\phi\|^{1/2}_{L^1} \|f\|_{L^2(v)}$$

where $A = \min(A_1, A_2)$ and A_1 and A_2 are given by

$$\begin{aligned} A_1^2 &= \sup_{\xi \in \mathbf{R}^n} \int |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)}\, d\eta \\ A_2^2 &= \sup_{\eta \in \mathbf{R}^n} \int |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)}\, d\xi. \end{aligned}$$

Proof. It is easy to see that the boundedness of $T : L^2(v) \to L^2(w)$ is equivalent to showing that $S : L^2 \to L^2$ where S is the map

$$Sf(\xi) = \int_{\mathbf{R}^n} w(\xi)^{1/2} \phi(\xi - \eta) v(\eta)^{-1/2} f(\eta)\, d\eta.$$

In addition, the norm of T on the weighted spaces is equal to the norm of S on the unweighted space.

To estimate $\|Sf\|_{L^2}$, we use the Cauchy-Schwarz inequality and then Tonelli to obtain

$$\begin{aligned} \|Sf\|^2_{L^2} &\leq \int_{\mathbf{R}^n} |\int_{\mathbf{R}^n} \phi(\xi - \eta) v(\eta)^{-1/2}\, d\eta|^2 w(\xi)\, d\xi \\ &\leq \int_{\mathbf{R}^n} |f(\eta)|^2 \int |\phi(\xi - \eta)| \frac{w(\xi)}{v(\eta)}\, d\xi\, d\eta \int_{\mathbf{R}^n} |\phi(\lambda)|\, d\lambda. \end{aligned}$$

Thus, we have that the operator norm $\|S\|_{\mathcal{L}(L^2)}$ is majorized by $\|\phi\|_{L^1}A_2$. An identical argument gives that the norm of S^* is controlled by $\|\phi\|_{L^1}^{1/2}A_1$. Since $\|S\|_{\mathcal{L}(L^2)} = \|S^*\|_{\mathcal{L}(L^2)} = \|T\|_{\mathcal{L}(L^2(v),L^2(w))}$, the Lemma follows.

∎

Our next Lemma shows that functions in the space $\dot{X}_\zeta^{1/2}$ are locally in L^2. For this Lemma, we introduce operators H and L which give the high and low-frequency parts of u. We let ϕ be smooth a function which is 1 for $|\xi| < 1$, 0 for $|\xi| > 2$. Given ζ, we define $Lf = (\phi(\cdot/16|\zeta|)\hat{f})\check{}$ and $Hf = f - Lf$.

We note that the operator L has the property that

$$\|\frac{\partial^\alpha}{\partial x^\alpha} Lf\|_{L^2(\mathbf{R}^n)} \le C|\zeta|^{|\alpha|}\|f\|_{L^2(\mathbf{R}^n)}. \tag{12.2}$$

Lemma 12.3 *Let ψ be a Schwartz function and suppose that $\zeta \in \mathcal{V}$, with $|\zeta| > 1$. then we have*

$$\begin{aligned} \|\psi u\|_{\dot{X}_\zeta^{-1/2}} &\le C\|u\|_{X_\zeta^{-1/2}} && (12.4)\\ \|\psi u\|_{X_\zeta^{1/2}} &\le C\|u\|_{\dot{X}_\zeta^{1/2}} && (12.5)\\ \|\psi u\|_{L^2} &\le C|\zeta|^{-1/2}\|u\|_{\dot{X}_\zeta^{1/2}} && (12.6)\\ \|\nabla H(\psi u)\|_{L^2} &\le C\|u\|_{\dot{X}_\zeta^{1/2}} && (12.7)\\ \|H(\psi u)\|_{L^2} &\le C|\zeta|^{-1}\|u\|_{\dot{X}_\zeta^{1/2}} && (12.8)\end{aligned}$$

The constant depends on n and ψ but is independent of the dimension.

Proof. If we take the Fourier transform of $u\psi$, we obtain the convolution of $\hat{u}$ and $\hat{\psi}$. Thus (12.4) will follow from Lemma 12.1, if we can show that

$$\int_{\mathbf{R}^n} |\hat{\psi}(\xi-\eta)|\frac{|p_\zeta(\eta)| + |\zeta|}{|p_\zeta(\xi)|}\, d\xi \le C. \tag{12.9}$$

To establish (12.9), we write $p_\zeta(\eta) = -|\eta-\xi+\xi|^2 + 2i(\eta-\xi+\xi) = -|\eta-\xi|^2 - 2\xi\cdot(\eta-\xi) + 2i\zeta\cdot(\eta-\xi) + p_\zeta(\xi)$ and thus we have expression in (12.9) is controlled by

$$\int_{\mathbf{R}^n} |\hat{\psi}(\xi-\eta)|(|\frac{|\zeta| + |\xi-\eta|^2 + |\zeta||\xi-\eta|}{|p_\zeta(\xi)|} + \frac{|\xi||\xi-\eta|}{|p_\zeta(\xi)|} + 1)\, d\xi. \tag{12.10}$$

We fix η and we will establish the estimate

$$\int_{\mathbf{R}^n} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi \le C. \tag{12.11}$$

The remaining terms in (12.10) are handled in a similar manner.

To establish (12.11), we let $B_k = \{\xi : |\xi-\eta| < 2^k\}$ then put $A_k = B_k \setminus B_{k-1}$. Using Lemma 9.3 and that $\hat{\psi}$ is in $\mathcal{S}(\mathbf{R}^n)$, we obtain for any N

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi &\le \int_{B_0} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi + \sum_{k=1}^{\infty}\int_{A_k} \frac{|\hat{\psi}(\xi-\eta)||\zeta|}{|p_\zeta(\xi)|}\,d\xi \\ &\le C\sum_{k=0}^{\infty} 2^{k((n-1)-N)}.\end{aligned}$$

If we choose $N > n-1$, we obtain the desired bound.

We turn to the proof of the remaining estimates. The estimate (12.5) is the transpose of (12.4). The adjoint of the map $Tu = \psi u$ is $T^*u = \bar{\psi}u$. As we know that $T^* : X_\zeta^{-1/2} \to \dot{X}_\zeta^{-1/2}$, it follows that $T : \dot{X}_\zeta^{1/2} \to X_\zeta^{1/2}$. The estimate (12.6) follows from (12.5) since we have $|\zeta|^{1/2} \le (|\zeta| + |p_\zeta(\xi)|)^{1/2}$.

Estimates (12.7-12.8) follow easily if we recall the definition of H and observe that $|p_\zeta(\xi)| \ge \frac{1}{2}|\xi|^2$ if $|\xi| > 4|\zeta|$. ∎

12.2 Estimates for potentials of negative order

Given a conductivity γ which continuously differentiable and is one near infinity, we may define a map $m_q : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ by

$$m_q(u)(v) = -\int \nabla\sqrt{\gamma}\cdot\nabla(\frac{uv}{\sqrt{\gamma}})\,dx.$$

When γ has two derivatives, it is easy to see that $m_q(u)$ is the function qu where $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. It is clear that the expression for $m_q(u)(v)$ can be extended to functions u and v which lie in the Sobolev space $L^{2,1}_{loc}$. Our next step is to study the expression $m_q(u)(v)$ when u and v lie in $\dot{X}_\zeta^{1/2}$. Note that because we assume that γ is one near infinity, it follows that q is compactly supported. Thus if ψ is a function which is one on a neighborhood of the support of q, we have $m_q(u)(v) = m_q(\psi u)(\psi v)$. This observation allows us to apply the estimates of Lemma 12.3.

We introduce the Hölder or Lipschitz spaces, $W^{s,\infty}(\mathbf{R}^n)$. If $s \geq 0$, we write $s = k + \theta$ with $0 < \theta \leq 1$. The space of Hölder continuous functions of order s is the collection of functions u so that $\frac{\partial^\alpha}{\partial x^\alpha} u \in L^\infty(\mathbf{R}^n)$ for $|\alpha| \leq k$ the derivatives of order k are *Hölder continuous* (or Lipschitz) with exponent θ. Thus, if

$$[u]_\theta = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\theta}$$

a function u is in $W^{s,\infty}(\mathbf{R}^n)$ if the norm defined below is finite

$$\|u\|_{W^{s,\infty}(\mathbf{R}^n)} = \sum_{|\alpha| \leq k} \|\frac{\partial^\alpha u}{\partial x^\alpha}\|_{L^\infty(\mathbf{R}^n)} + \sum_{|\alpha| = k} [\frac{\partial^\alpha u}{\partial x^\alpha}]_\theta.$$

Our main result is the following Theorem.

Theorem 12.12 *Let $\zeta_j \in \mathcal{V}$, for $j = 1, 2$ and suppose that $|\zeta_1| = |\zeta_2|$ and suppose $\theta \in [0, 1]$. If $\gamma \in W^{1+\theta,\infty}(\mathbf{R}^n)$, $\nabla\gamma$ is compactly supported, and $\gamma \geq c$ for some $c > 0$, then we have*

$$|m_q(u)(v)| \leq C\omega(\|\log\gamma\|_{W^{1+\theta,\infty}(\mathbf{R}^n)})|\zeta_1|^{-\theta}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.$$

where the function ω satisfies $\lim_{t\to 0^+} \omega(t) = 0$.

In addition, if $\theta = 0$, we have

$$|m_q(u)(v)| \leq C\omega(\gamma, |\zeta|)\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.$$

where $\lim_{s\to\infty} \omega(\gamma, s) \leq \lim_{\epsilon\to 0^+} \|\log\gamma - (\log\gamma)_\epsilon\|_{W^{1,\infty}(\mathbf{R}^n)}$.

Exercise 12.13 *Let ϕ be a standard mollifier. Thus ϕ is supported in a ball of radius 1, $\int_{B_1(0)} \phi\, dy = 1$, and $\phi \in \mathcal{D}(\mathbf{R}^n)$. Set $\phi_\epsilon = \epsilon^{-n}\phi_\epsilon$ for $\epsilon > 0$. Let θ be in $(0, 1]$. Prove that for α with $|\alpha| \geq 1$, we have*

$$\epsilon^{-\theta}\|u - u_\epsilon\|_{L^\infty(\mathbf{R}^n)} + \epsilon^{|\alpha|-\theta}\|\frac{\partial^\alpha}{\partial x^\alpha} u_\epsilon\|_{L^\infty(\mathbf{R}^n)} \leq C(\alpha, \phi)[u]_\theta.$$

Again, let $\theta \in (0, 1]$. Conversely, show that for any locally integrable function u, after modifying u on a set of measure we have

$$[u]_\theta \leq \sup_{\epsilon > 0}(\epsilon^{-\theta}\|u - u_\epsilon\|_{L^\infty(\mathbf{R}^n)} + \epsilon^{1-\theta}\|\nabla u_\epsilon\|_{L^\infty(\mathbf{R}^n)}).$$

Exercise 12.14 *Let ϕ be a standard mollifier, set $\phi_\epsilon(x) = \epsilon^{-n}(x/\epsilon)$ and f locally integrable, define $f_\epsilon = \phi_\epsilon * f$. Define*

$$d = \lim_{\epsilon \to 0^+} \|f - f_\epsilon\|_{L^\infty}.$$

Suppose that f is compactly supported in $\mathbf{R}^n$. Show that d is the distance in L^∞ from f to the continous functions. That is show that

$$d = \inf\{\|f - g\|_{L^\infty(\mathbf{R}^n)} : g \in C(\mathbf{R}^n)\}.$$

Our Theorem follows easily from the following Lemma.

Lemma 12.15 *Let ζ_1 and ζ_2 lie in $\mathcal{V}$ and $|\zeta_1| = |\zeta_2|$, suppose $\theta \in [0,1]$ and let f be a vector-valued function in $W^{\theta,\infty}(\mathbf{R}^n)$. Let ψ be a Schwartz function and put $u_\psi = \psi u$. Then we have*

$$\int f \cdot \nabla(u_\psi v_\psi) \leq C\|f\|_{L^\infty(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}} \tag{12.16}$$

$$\int f \cdot \nabla(u_\psi v_\psi) \leq Cs^{-1}\|\nabla f\|_{L^\infty(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}} \tag{12.17}$$

$$\int f \cdot \nabla(u_\psi v_\psi) \leq Cs^{-\theta}\|f\|_{W^{\theta,\infty}(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}} \tag{12.18}$$

The constants depend on n, θ, and ψ.

Proof. To prove (12.17), we integrate parts and use estimate (12.6) from Lemma 12.3.

To prove (12.16), we write $u_\psi = Hu_\psi + Lu_\psi$ and $v_\psi = Hv_\psi + Lv_\psi$ and then

$$\int_{\mathbf{R}^n} f \cdot \nabla(u_\psi v_\psi)\,dy = \int f \cdot \nabla(Hu_\psi Hv_\psi + Hu_\psi Lv_\psi + Lu_\psi Hv_\psi + Lu_\psi Lv_\psi)\,dy.$$

Thus, we have four terms to consider. For the term involving the product of high-frequency terms, we use the Leibniz rule, the Cauchy-Schwarz inequality, and estimates (12.7) and (12.8) to obtain

$$|\int_{\mathbf{R}^n} f \cdot \nabla(Hu_\psi Hv_\psi)\,dy| \leq Cs^{-1}\|f\|_{L^\infty(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.$$

To estimate the mixed terms involving $Hu_\psi Lv_\psi$, we use the Leibniz rule, the Cauchy-Schwarz inequality, and the estimates (12.2), (12.5), (12.7), and (12.8) to obtain

$$\begin{aligned} &|\int_{\mathbf{R}^n} f \cdot (Lv_\psi \nabla Hu_\psi + Hu_\psi \nabla Lv_\psi)\,dy| \\ &\qquad \leq C\|f\|_{L^\infty(\mathbf{R}^n)}(\|\nabla Hu_\psi\|_{L^2(\mathbf{R}^n)}\|Lv_\psi\|_{L^2(\mathbf{R}^n)} + \|Hu_\psi\|_{L^2(\mathbf{R}^n)}\|\nabla Lv_\psi\|_{L^2(\mathbf{R}^n)} \\ &\qquad\qquad \leq C\|f\|_{L^\infty(\mathbf{R}^n)})\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}. \end{aligned}$$

Of course, the mixed term involving $Lu_\psi Hv_\psi$ can be handled by permuting u and v. Finally, for the term involving $Lu_\psi Lv_\psi$, we observe that the Leibniz rule and estimate (12.2) give

$$\|\nabla(Lu_\psi Lv_\psi)\|_{L^1(\mathbf{R}^n)} \leq C|\zeta|\|Lu_\psi\|_{L^2(\mathbf{R}^n)}\|Lv_\psi\|_{L^2(\mathbf{R}^n)}.$$

Now the estimate (12.6) gives

$$\|\nabla(Lu_\psi Lv_\psi)\|_{L^1(\mathbf{R}^n)} \leq C\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.$$

We turn to the proof of the third estimate (12.18). This follows by a simple interpolation argument. We let ϕ_ϵ be a standard mollifier and set $f_\epsilon = \phi_\epsilon * f$. We observe that for $0 \leq \theta \leq 1$, we have $\|f - f_\epsilon\|_{L^\infty(\mathbf{R}^n)} \leq C\epsilon^\theta\|f\|_{W^{\theta,\infty}(\mathbf{R}^n)}$ and $\|\nabla f_\epsilon\|_{L^\infty(\mathbf{R}^n)} \leq C\epsilon^{\theta-1}\|f\|_{W^{\theta,\infty}(\mathbf{R}^n)}$. Thus, using estimates (12.16) and (12.17), we have

$$\begin{aligned}\int f \cdot \nabla(u_\phi v_\psi)\, dy &= \int (f - f_\epsilon)\cdot\nabla(u_\phi v_\psi)\, dy + \int f_\epsilon \cdot \nabla(u_\phi v_\psi)\, dy \\ &\leq C(\epsilon^\theta + |\zeta|^{-1}\epsilon^{\theta-1})\|f\|_{W^{\theta,\infty}(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.\end{aligned}$$

Choosing $\epsilon = 1/s$ gives (12.18). ∎

We now give the proof of the estimates of Theorem 12.12.

Proof of Theorem 12.12. We choose a function $\psi \in \mathcal{D}(\mathbf{R}^n)$ which is one on a neighborhood of the support $\nabla\gamma$. With $u_\psi = \psi u$, we have

$$m_q(u)(v) = -\int_{\mathbf{R}^n} \frac{\nabla\sqrt{\gamma}}{\sqrt{\gamma}}\cdot\nabla(u_\psi v_\psi) - \frac{|\nabla\sqrt{\gamma}|^2}{\gamma}u_\psi v_\psi\, dy$$

From the estimate (12.6), it is easy to see that

$$|\int \frac{|\nabla\sqrt{\gamma}|^2}{\gamma}u_\psi v_\psi\, dy| \leq C|\zeta_1|^{-1}\|\log\sqrt{\gamma}\|^2_{W^{1,\infty}(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}$$

which is stronger than the conclusion of the Theorem. For the other term in $m_q(u)(v)$, we obtain the estimate

$$|\int_{\mathbf{R}^n} \nabla\log\sqrt{\gamma}\cdot\nabla(u_\psi v_\psi)\, dy| \leq C|\zeta|^{-\theta}\|\log\sqrt{\gamma}\|_{W^{1+\theta,\infty}(\mathbf{R}^n)}\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.$$

Finally, when $\log\sqrt{\gamma}$ is only Lipschitz (and compactly supported), we let $(\log\sqrt{\gamma})_\epsilon$ denote a standard mollification of $\log\sqrt{\gamma}$. We add and subtract the smoothed out version of $(\log \sqrt{\gamma})_\epsilon$, integrate by parts in the second term and use the estimates (12.18) and (12.6) to obtain

$$
\begin{aligned}
&|\int_{\mathbf{R}^n} \nabla \log \sqrt{\gamma} \cdot \nabla(u_\psi v_\psi)\, dy| \\
&\qquad \le |\int_{\mathbf{R}^n} \nabla(\log\sqrt{\gamma} - (\log\sqrt{\gamma})_\epsilon) \cdot \nabla(u_\psi v_\psi)\, dy| + |\int_{\mathbf{R}^n} \Delta(\log(\sqrt{\gamma})_\epsilon) \cdot (u_\psi v_\psi)\, dy| \\
&\le C(\|\nabla(\log\sqrt{\gamma} - (\log\sqrt{\gamma})_\epsilon)\|_{L^\infty(\mathbf{R}^n)} + \epsilon^{-1}|\zeta|^{-1}\|\Delta(\log\sqrt{\gamma})_\epsilon\|_{L^\infty(\mathbf{R}^n)})\|u\|_{\dot{X}^{1/2}_{\zeta_1}}\|v\|_{\dot{X}^{1/2}_{\zeta_2}}.
\end{aligned}
$$

If we choose $\epsilon = |\zeta|^{-1/2}$, then we obtain the last conclusion of the theorem. ∎

12.3 An averaged estimate

We will construct solutions of the Schrödinger equation in the form $\exp(x \cdot \zeta)(1 + \psi)$ where $\zeta \in \mathcal{V}$. The function ψ will satisfy $\psi - G_\zeta(q\psi) = G_\zeta(q)$ with $G_\zeta = (\Delta + 2\zeta \cdot \nabla)^{-1}$. It is trivial to see that $G_\zeta : \dot{X}^{-1/2}_\zeta \to \dot{X}^{1/2}_\zeta$ and Theorem 12.12 tells us that m_q maps $\dot{X}^{1/2}_\zeta \to \dot{X}^{-1/2}_\zeta$. It remains to study the right-hand side, $G_\zeta(q)$. The new ingredient in Tataru-Haberman's argument is the following averaged estimate for this function.

Before stating the main lemma, we introduce some notation. For the remainder on this chapter, we fix a vector $k \in \mathbf{R}^n$ and let P be a plane that is perpendicular to k (or P may be an arbitrary plane if $k = 0$). We let e_1 and e_2 be an orthonormal basis for P and define

$$
\begin{aligned}
e_1(\theta) &= e_1 \cos(\theta) - e_2 \cos(\theta) \\
e_2(\theta) &= e_1 \sin(\theta) + e_2 \cos(\theta)
\end{aligned}
$$

For $s > |k|$, we let

$$
\begin{aligned}
\zeta_1(s,\theta) &= se_1(\theta) + i(\frac{k}{2} + \sqrt{s^2 - |k|^2/4}e_2(\theta) \\
\zeta_2(s,\theta) &= -se_1(\theta) + i(\frac{k}{2} - \sqrt{s^2 - |k|^2/4}e_2(\theta)
\end{aligned}
$$

Lemma 12.19 *Let $\lambda > |k|$ and fix $\psi \in \mathcal{S}(\mathbf{R}^n)$. If $f \in L^{2,\theta}(\mathbf{R}^n)$, $0 \le \theta \le 1$, then for $j = 1$ or 2, we have*

$$
\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\psi\nabla f\|^2_{\dot{X}^{-1/2}_{\zeta_j}}\, ds\, d\theta \le C(\psi,k)\lambda^{-\theta}\|f\|^2_{L^{2,\theta}}. \tag{12.20}
$$

If $f \in L^2$, then

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\psi\nabla f\|^2_{\dot{X}_\zeta^{-1/2}}\,ds\,d\theta = o(1), \qquad \text{as } \lambda \to \infty.$$

Proof. We let $\zeta_1 = \zeta$ and give the proof in detail for ζ. The proof for ζ_2 can be carried by replacing the basis $\{e_1, e_2\}$ by $\{-e_1, -e_2\}$. We claim the following two estimates.

$$\int_0^{2\pi}\int_\lambda^{2\lambda} \|\psi\nabla f\|^2_{\dot{X}_\zeta^{-1/2}}\,ds\,d\theta \;\leq\; C\|\nabla f\|^2_{L^2} \tag{12.21}$$

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\psi\nabla f\|^2_{\dot{X}_\zeta^{-1/2}}\,ds\,d\theta \;\leq\; C\|f\|^2_{L^2} \tag{12.22}$$

To establish (12.21), we use (12.4) and the elementary estimate $\|\nabla f\|^2_{X_\zeta^{-1/2}} \leq |\zeta|^{-1}\|\nabla f\|^2_{L^2}$ and then integrate in s and λ.

The second estimate (12.22) is more interesting because it is not proved by integrating a pointwise estimate. The average is substantially smaller than maximum. To establish, (12.22), we begin by using (12.4) to replace the homogeneous space $\dot{X}_\zeta^{-1/2}$ by the inhomogeneous space $X_\zeta^{-1/2}$, then we use Plancherel's theorem and divide the integral on $\mathbf{R}^n$ into three regions,

$$\begin{aligned}
\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\psi\nabla f\|^2_{\dot{X}_\zeta^{-1/2}}\,ds\,d\theta &\leq C\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda} \|\nabla f\|^2_{X_\zeta^{-1/2}}\,ds\,d\theta \\
&\leq \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|<16|k|} \frac{|\xi|^2}{|p_\zeta(\xi)|+|\zeta|}|\hat{f}(\xi)|^2\,d\xi\,dsd\theta \\
&\quad + \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|^2<16\lambda|\xi^\perp|} \frac{|\xi|^2}{|p_\zeta(\xi)|+|\zeta|}|\hat{f}(\xi)|^2\,d\xi\,ds\,d\theta \\
&\quad + \frac{C}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\int_{|\xi|^2>16\lambda|\xi^\perp|,|\xi|>16|k|} \frac{|\xi|^2}{|p_\zeta(\xi)|+|\zeta|}|\hat{f}(\xi)|^2\,d\xi\,ds\,d\theta \\
&= I + II + III
\end{aligned} \tag{12.23}$$

We use $\xi^\perp$ denote the projection of ξ onto the plane P. Note that $|\xi^\perp|^2 = |e_1(\theta)\cdot\xi|^2 + |e_2(\theta)\cdot\xi|^2$.

In the region $|\xi| < 16|k|$, we have

$$I \leq C\frac{|k|^2}{|\zeta|}\|f\|^2_{L^2}$$

which is sufficient since our constants may depend on k.

To estimate II, we will have to work harder. We change variables in the (s,θ) integral by setting $t_1 = \operatorname{Re} p_\zeta(\xi) = -|\xi|^2 - k\cdot\xi - \sqrt{s^2-|k|^2/4}\, e_2(\theta)\cdot\xi$ and $t_2 = \operatorname{Im} p_\zeta(\xi) = 2se_1(\theta)\cdot\xi$. It is easy to compute J, the Jacobian of the map $(s,\theta)\to t$ as

$$\begin{aligned} J &= \left| \begin{array}{cc} \frac{2s}{\sqrt{s^2-|k|^2/4}}\, e_2(\theta)\cdot\xi & 2e_1(\theta)\cdot\xi \\ 2\sqrt{s^2-|k|^2/4}\, e_1(\theta)\cdot\xi & -2se_2(\theta) \end{array} \right| \\ &= 4\sqrt{s^2-|k|^2/4}|e_1(\theta)\cdot\xi|^2 + \frac{4s^2}{\sqrt{s^2-|k|^2/4}}|e_2(\theta)\cdot\xi|^2. \end{aligned}$$

Under our assumption that $s\ge\lambda\ge|k|$, we have that $2s\sqrt{3}|\xi^\perp|^2 \le J \le (8s/\sqrt{3})|\xi^\perp|^2$. We also have that the image of $(\lambda,2\lambda)\times(0,2\pi)$ under the map $(s,\theta)\to(t_1+it_2)$ lies in a disk of radius $4\lambda|\xi^\perp|$. Thus, we have

$$II \le C\int_{|\xi|^2<16\lambda|\xi^\perp|} |\hat f(\xi)|^2 \frac{|\xi|^2}{\lambda^2|\xi^\perp|^2}\int_{|t|<4\lambda|\xi^\perp|}\frac{1}{|t|}\,dt\,d\xi \le C\|f\|^2_{L^2}.$$

The integral in t is taken over a disk centered at the origin as this disk gives the largest value. This calculation gives the desired bound in the region $\{\xi : |\xi|^2 \le 16\lambda|\xi^\perp|\}$. Finally, when $|\xi| > 16|k|$ and $|\xi|^2 > 16\lambda|\xi^\perp|$ we have $|p_\zeta(\xi)| \ge \frac{1}{2}|\xi|^2 + |\xi|(\frac{1}{4}|\xi| - |k|) + (\frac{1}{4}|\xi|^2 - 4\lambda|\xi^\perp|) \ge \frac{1}{2}|\xi|^2$ and thus it is easy to see that $III \le C\|f\|^2_{L^2}$.

The first conclusion of the theorem follows by interpolating between (12.21) and (12.22). See (4.9). To prove the second we write $f = f_\epsilon + (f - f_\epsilon)$ where f_ϵ is a standard mollification of f. Then from (12.21) and (12.22), we obtain

$$\frac{1}{\lambda}\int_0^{2\pi}\int_\lambda^{2\lambda}\|\psi\nabla f\|^2_{\dot X_\zeta^{-1/2}}\,ds\,d\theta \le C(\lambda^{-1}\|\nabla f_\epsilon\|^2_{L^2} + \|f-f_\epsilon\|_{L^2}).$$

If we let $\epsilon = \lambda^{-1/2}$, the right-hand side goes to zero. ∎

Theorem 12.24 *Let γ satisfy (11.7), suppose that $\gamma = 1$ outside some compact set and that $\nabla\gamma$ is in $L^2(\mathbf{R}^n)$. Let k and $\zeta_j(s,\theta)$ be as in Lemma previous result. Then we have*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_\lambda^{2\lambda}\int_0^{2\pi}\|\Delta\sqrt{\gamma}/\sqrt{\gamma}\|^2_{\dot X_{\zeta_j}^{-1/2}}\,d\theta\,ds = 0.$$

Proof. We may write $\Delta\sqrt{\gamma}/\sqrt{\gamma}$ as $\operatorname{div}(\nabla\log\sqrt{\gamma}) + |\nabla\log\sqrt{\gamma}|^2$. We may estimate the average of the $\dot X_{\zeta_\ell}^{-1/2}$ norm for the first term by Lemma 12.19. As $|\nabla\log\sqrt{\gamma}|^2$ is in $L^2(\mathbf{R}^n)$, we may use (12.4) and the elementary inequality $(|\zeta| + |p_\zeta(\xi)|)^{-1} \le |\zeta|^{-1}$ to conclude that $\|\psi|\nabla\log\sqrt{\gamma}|^2\|_{\dot X_{\zeta_\ell}^{-1/2}} \le |\zeta|^{-1/2}\|\nabla\log\sqrt{\gamma}\|_{L^2(\mathbf{R}^n)}$. We may average this estimate to obtain the desired limit.

∎

12.4 Notes

The results of this chapter are taken from the work of Haberman and Tataru [15]. The inverse conductivity problem or Calderón problem was introduced to the mathematics community by A.P. Calderón in [10]. Sylvester and Uhlmann's groundbreaking paper [36] constructed the complex geometrical optics solutions and showed how they could be used to solve the inverse conductivity problem. The work of Brown [7], Brown and Torres [8], and Panchenko, Päivärinta and Uhlmann [24] showed that the method of Sylvester and Uhlmann could be used when the conductivity had 3/2 derivatives.

Y. Zhang helped to prepare the material in this chapter.

Chapter 13

Inverse Problems: Global uniqueness for C^1 conductivities

In this chapter, we assemble the results of the past few chapters to prove Haberman and Tataru's version of Sylvester and Uhlmann's theorem [36].

We begin with an extension result.

Proposition 13.1 *Let Ω be a C^1-domain. If $\gamma \in C^1(\bar{\Omega})$ is a conductivity satisfying (11.7), there exists an extension $E\gamma \in C^1(\mathbf{R}^n)$, $E\gamma$ satisfies (11.7), and $E\gamma = 1$ outside some compact set. In addition, the values of $E\gamma$ in $\mathbf{R}^n \setminus \Omega$ depend only on the boundary values of γ and $\nabla\gamma$.*

Remark. This may also be obtained as consequence of the Whitney extension theorem. The Whitney extension theorems holds in a much more general class of domains.

Proof. We will extend $\log\gamma$ to give a function which is compactly supported in $\mathbf{R}^n$. Taking the exponential gives the desired extension of γ.

Let $\mathbf{R}^n_=\{x : x_n < 0\}$. Using a partition of unity and a change of variables, it suffices to extend a function $\phi \in C^1(\bar{\mathbf{R}}^n_-)$ and is zero for x such that $|x| \geq 1$ to a function which is C^1 and compactly supported in $\mathbf{R}^n$.

To define the extension, we let η be a standard mollifier on $\mathbf{R}^{n-1}$ and let $\eta_{x_n}(x') = x_n^{1-n}\eta(x'/x_n)$ and we define $E_1\phi$ by

$$E_1\phi(x', x_n) = \begin{cases} \phi(x', x_n), & x_n \leq 0 \\ \phi(x', 0) + x_n(\eta_{x_n} * \frac{\partial\phi}{\partial x_n}(\cdot, 0))(x'), & x_n > 0 \end{cases}$$

Note that the convolution in the definition of $E_1\phi$ is a convolution on $\mathbf{R}^{n-1}$. We put $E\phi = \psi E_1\phi$ where ψ is in $\mathcal{D}(\mathbf{R}^n)$, $\psi = 1$ on $B_2(0)$ and ψ is supported in $B_4(0)$.

It remains to show that $E_1\phi$ is C^1. Thus, we need to show that the derivatives of $E_1\phi$ are continuous at $x_n = 0$. We compute the derivative with respect to x_i, $i = 1, \dots, n-1$ for $x_n > 0$.

$$\frac{\partial}{\partial x_i} E_1\phi(x) = \frac{\partial \phi}{\partial x_i}(x', 0) + (\frac{\partial \eta}{\partial x_i})_{x_n} * \frac{\partial \phi}{\partial x_n}(\cdot, 0))(x')$$

Since the $\partial\eta/\partial x_n$ has mean value zero, we have that

$$\lim_{x\to 0}((\frac{\partial \eta}{\partial x_i})_{x_n} * \frac{\partial \phi}{\partial x_n}(\cdot, 0))(x') = 0$$

Next, we compute the derivative of $E_1\phi$ with respect to x_n and obtain

$$\frac{\partial}{\partial x_n} E_1\phi(x', x_n) = (\eta_{x_n} * \frac{\partial \phi}{\partial x_n}(\cdot, x_n))(x') + (\psi_{x_n} * \frac{\partial \phi}{\partial x_n}(\cdot, 0))(x')$$

where $\psi(x') = (1-n)\eta(x') + \sum_{i=1}^{n-1} x_i \frac{\partial \eta}{\partial x_i}(x')$. The integral of ψ is given by

$$\int_{\mathbf{R}^{n-1}} \psi \, dx' = \frac{\partial}{\partial x_n}\bigg|_{x_n=1} \int_{\mathbf{R}^{n-1}} \eta_{x_n}(x') \, dx' = 0.$$

Thus we have that the derivative $\partial E_1\phi/\partial x_n$ is continuous at 0. ∎

Lemma 13.2 *Let $n \geq 3$, $k \in \mathbf{R}^n$, and for $l = 1, 2$ suppose that γ_ℓ satisfies (11.7), $\nabla\gamma_\ell \in L^2(\mathbf{R}^n)$ and $\gamma_1 = \gamma_2 = 1$ outside some compact set. Then we may find sequences $\{\zeta_\ell^j\}_{j=1}^\infty \subset \mathcal{V}$, $\ell = 1, 2$ so that $\zeta_1^j + \zeta_2^j = ik$ and*

$$\lim_{j\to\infty} \|q_\ell\|_{\dot{X}^{-1/2}_{\zeta_\ell^j}} = 0.$$

Proof. The lemma follows immediately from Theorem 12.24. ∎

Lemma 13.3 *If $\gamma_\ell \in C^1(\mathbf{R}^n)$, γ_ℓ satisfies (11.7) and $\gamma_\ell = 1$ outside a compact set. Let $\{\zeta_\ell^j\}_{j=1}^\infty$ be as in the previous Lemma. We may find $\psi_\ell^j \in \dot{X}^{1/2}_{\zeta_\ell^j}$ so that $v_\ell = e^{x\cdot\zeta_\ell^j}(1 + \psi_\ell^j)$ satisfies $\Delta v - m_{q_\ell}(v) = 0$ and*

$$\lim_{j\to\infty} \|\psi_\ell^j\|_{\dot{X}^{1/2}_{\zeta_\ell^j}} = 0.$$

In addition, we have that $\psi_\ell^j \in L^{2,1}_{loc}(\mathbf{R}^n)$ and that

$$\int \nabla v_\ell \cdot \nabla\phi \, dy + m_{q_\ell}(v)(\phi) = 0, \qquad \phi \in \mathcal{D}(\mathbf{R}^n). \tag{13.4}$$

Remark. If a function v is in $L^{2,1}_{loc}$ and satisfies (13.4), we will say that v is *a weak solution* of the equation $(\Delta - q)v = 0$.

Proof. From the definition of the $\dot X^b_\zeta$ norms, we have $\|G_\zeta f\|_{\dot X^{1/2}_\zeta} = \|f\|_{\dot X^{-1/2}_\zeta}$ and by Theorem 12.12, we have $\lim_{\zeta\to\infty} \|m_q\|_{\mathcal{L}(\dot X^{1/2}_\zeta, \dot X^{-1/2}_\zeta)} = 0$. Thus, we may find R so that $\|G_\zeta \circ m_{q_\ell}\|_{\mathcal{L}(\dot X^{1/2}_\zeta)} \le 1/2$ if $|\zeta| > R$. For such ζ, we have that $(I - G_\zeta \circ m_q)^{-1}$ exists and is given by the series $\sum_{j=0}^\infty (G_\zeta \circ m_q)^j$. This series converges absolutely in the operator norm on $\mathcal{L}(\dot X^{1/2}_\zeta)$ and we have $\|(I - G_\zeta \circ m_{q_\ell}\|_{\mathcal{L}(\dot X^{1/2}_\zeta)} \le 2$.

Now we let $\{\zeta^j_\ell\}_{j=1}^\infty$ be the sequences from Lemma 13.2. For j sufficiently large, we set

$$\psi^j_\ell = (I - G_{\zeta^j_\ell} \circ m_{q_\ell})^{-1}(G_{\zeta^j_\ell} q_\ell).$$

By the observations of the previous paragraph,

$$\lim_{j\to\infty} \|\psi^j_\ell\|_{\dot X^{1/2}_{\zeta^j_\ell}} \le 2 \lim_{j\to\infty} \|q_\ell\|_{\dot X^{-1/2}_{\zeta^j_\ell}} = 0.$$

It is straightforward to show that the corresponding v_ℓ is a solution of $(\Delta - q_\ell)v_\ell = 0$ in the sense of distributions. From (12.2) and (12.7), it follows that $\nabla\psi^j_\ell$ and hence ∇v_ℓ is in $L^2_{loc}(\mathbf{R}^n)$. Hence, we may obtain the weak formulation of the equation $(\Delta - q_\ell)v_\ell = 0$ given in (13.4). ∎

Lemma 13.5 *Suppose γ_1 and γ_2 are two C^1-conductivities, $\gamma_1 = \gamma_2$ outside Ω and $\gamma_1 = \gamma_2 = 1$ outside a compact set and assume $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. If $v_\ell \in L^{2,1}_{loc}(\mathbf{R}^n)$, $\ell = 1, 2$, are solutions of $(\Delta - q_\ell)v_\ell = 0$ in $\mathbf{R}^n$, then we have*

$$m_{q_1}(v_1)(v_2) = m_{q_2}(v_2)(v_1). \tag{13.6}$$

Proof. Let γ satisfy $\gamma \in C^1(\mathbf{R}^n)$ and (11.7). As usual, we let q denote the potential $\Delta\sqrt{\gamma}/\sqrt{\gamma}$. We first observe that if v is a weak solution of $(\Delta - q)v = 0$, then $u = v/\sqrt{\gamma}$ is a solution of $\operatorname{div}\gamma\nabla u = 0$. To see that ψ be a test function from $\mathcal{D}(\mathbf{R}^n)$ and let $v = \sqrt{\gamma}u$ and $\phi = \sqrt{\gamma}\psi$ and use the product rule to obtain

$$\begin{aligned}
&\int_{\mathbf{R}^n} \nabla(\sqrt{\gamma}u)\cdot\nabla(\sqrt{\gamma}\psi) - \nabla\sqrt{\gamma}\cdot\nabla(\sqrt{\gamma}u\psi)\,dy \\
&\quad = \int_{\mathbf{R}^n} \gamma\nabla u\cdot\nabla\psi + \sqrt{\gamma}\psi\nabla\sqrt{\gamma}\cdot\nabla u + u\nabla\sqrt{\gamma}\cdot\nabla(\sqrt{\gamma}\psi) - \nabla\sqrt{\gamma}\cdot\nabla(\sqrt{\gamma}u\psi)\,dy
\end{aligned}$$

Since the map $\psi \to \psi\sqrt{\gamma}$ is invertible on $C^1(\mathbf{R}^n)$, we have that

$$\int \nabla v\cdot\nabla\phi - \nabla\sqrt{\gamma}\cdot\nabla(v\phi/\sqrt{\gamma})\,dy = 0, \qquad \phi \in C^1_c(\mathbf{R}^n)$$

if and only if

$$\int_{\mathbf{R}^n} \gamma \nabla u \cdot \nabla \phi \, dy = 0, \qquad \phi \in C^1_c(\mathbf{R}^n).$$

We turn to the proof of (13.6). We first observe that since $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \Omega$, it is immediate that

$$\int_{\mathbf{R}^n\setminus\bar\Omega} \nabla\sqrt{\gamma_1} \cdot \nabla(v_1 v_2/\sqrt{\gamma_1})\, dy = \int_{\mathbf{R}^n\setminus\bar\Omega} \nabla\sqrt{\gamma_2} \cdot \nabla(v_1 v_2/\sqrt{\gamma_2})\, dy. \tag{13.7}$$

To obtain the corresponding result for the integral over Ω, we claim that

$$\Lambda_{\gamma_1}(u_1)(u_2) = \int_\Omega \nabla v_1 \cdot \nabla v_2 - \nabla\sqrt{\gamma}_1 \cdot \nabla(v_1 v_2/\sqrt{\gamma_1})\, dy.$$

We let $\tilde u_2 = v_2\sqrt{\gamma_1}$ then the product rule and the calculation in the previous paragraph give that

$$\int_\Omega \nabla v_1 \cdot \nabla v_2 - \nabla\sqrt{\gamma_1} \cdot \nabla(v_1 v_2/\sqrt{\gamma_1})\, dy = \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \tilde u_2 \, dy.$$

By the Lemma below $\tilde u_2 - u_2$ lies in $L^{2,1}_0(\Omega)$ and u_1 is a solution $\operatorname{div}\gamma_1\nabla u_1 = 0$ in Ω. Thus,

$$\int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \tilde u_2 \, dy = \Lambda_{\gamma_1}(u_1)(u_2).$$

If we switch 1 and 2, and use that $\Lambda_{\gamma_1}(u_1)(u_2) = \Lambda_{\gamma_2}(u_1)(u_2)$ we obtain that

$$\int_\Omega \nabla v_1 \cdot \nabla v_2 - \nabla\sqrt{\gamma_1} \cdot \nabla(v_1 v_2/\sqrt{\gamma_1})\, dy = \int_\Omega \nabla v_1 \cdot \nabla v_2 - \nabla\sqrt{\gamma_2} \cdot \nabla(v_1 v_2/\sqrt{\gamma_2})\, dy. \tag{13.8}$$

Adding (13.8) and (13.7) gives (13.6). ∎

The following technical result was used in the previous Lemma.

Lemma 13.9 *Suppose Ω is a C^1-domain. If $u \in L^{2,1}(\Omega)$, $\beta \in C^1(\bar\Omega)$ with $\beta = 0$ on $\partial\Omega$, then $\beta u \in L^{2,1}(\bar\Omega)$.*

Proof. From Lemma 10.10, we know that $C^\infty(\bar\Omega)$ is dense in $L^{2,1}(\Omega)$ and for $\beta \in C^1(\bar\Omega)$, we have that $u \to \beta u$ is a bounded map on $L^{2,1}(\Omega)$. Thus, it suffices to prove the Lemma when $u \in C^\infty(\bar\Omega)$.

Since $v = \beta u$ will be in $C^1(\bar\Omega)$ and v is 0 on $\partial\Omega$, it suffices to show that if v is a function in $C^1(\bar\Omega)$ and $v = 0$ on $\partial\Omega$, then $v \in L^{2,1}_0(\Omega)$.

Using a partition of unity and a change of variables, we may reduce to the case when v is supported in $\{x : |x| < 1,\ x_n \geq 0\}$. We let η be a function which is 1 if $x_n > 2$ and $x_n < 1$. We set $v_\epsilon(x) = \eta(x_n/\epsilon)v(x)$ and consider

$$\|v - \eta_\epsilon v\|^2_{L^{2,1}_0(\{x:x_n>0\})} = \int_{x_n>0} |1 - \eta(\cdot/\epsilon)|^2(|v|^2 + |\nabla v|^2) + \epsilon^{-2}|\eta'(\cdot/\epsilon)|^2|v(x)|^2\, dy.$$

The first term goes to zero with ϵ by the dominated convergence theorem. Since $v(x', 0) = 0$, the mean value theorem implies $|v(x', x_n)| \leq x_n\|\nabla v\|_{L^\infty}$. Since $\eta'(x_n/\epsilon)$ is supported in the strip $\{x : \epsilon < x_n < 2\epsilon\}$, the second term goes to zero with ϵ, too. Thus, we may approximate v by a sequence of compactly supported, C^1 functions. We may regularize as in Lemma 10.9 to obtain functions in $C^{\mathcal{D}}(\Omega)$ which converge to v. ∎

Theorem 13.10 *Let Ω be a C^1-domain, let γ_1 and γ_2 be conductivities in $C^1(\bar{\Omega})$ which satisfy (11.7). If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

Proof. According to Theorems 11.8 and 11.9, we have $\gamma_1 = \gamma_2$ and $\nabla\gamma_1 = \nabla\gamma_2$ on $\partial\Omega$. Thus, by Proposition 13.1, we may extend γ_1 and γ_2 to functions in $C^1(\mathbf{R}^n)$ which satisfy (11.7), $\gamma_1 = \gamma_2$ in $\mathbf{R}^n \setminus \bar{\Omega}$, and $\gamma_1 = \gamma_2 = 1$ outside a compact set.

Since the distributions $q_\ell = \Delta\sqrt{\gamma}_\ell/\sqrt{\gamma_\ell}$, are compactly supported we have that the Fourier transform $\hat{q}_\ell$ is given by the function $\hat{q}_\ell(e^{-ix\cdot k})$. We fix $k \in \mathbf{R}^n$ and show that $\hat{q}_1(k) = \hat{q}_2(k)$. According to Lemma we may find sequences $\{\zeta^j_\ell\} \subset \mathcal{V}$ and $\{\psi^j_\ell\}$ so that, $\zeta^j_1 + \zeta^j_2 = -ik$, $v^j_\ell = e^{x\cdot\zeta^j_\ell}(1 + \psi^j_\ell)$ are solutions of $\Delta v - qv = 0$ and $\lim_{j\to\infty} \|\psi^j_\ell\|_{X^{1/2}_{\zeta^j_\ell}} = 0$. It follows from Theorem 12.12 and Lemma 13.2 that

$$\lim_{j\to\infty} m_{q_\ell}(v^j_1)(v^j_2) = \hat{q}_\ell(ik).$$

Now (13.6) implies that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $q_1 = q_2$.

Finally, an observation of Alessandrini (I think) gives that if $q_1 = q_2$, then $\gamma_1 = \gamma_2$. The details are the next Lemma. ∎

Lemma 13.11 *If γ_1 and γ_2 in $C^1(\bar{\Omega})$ and if $\Delta\sqrt{\gamma}_1/\sqrt{\gamma}_1 = \Delta\sqrt{\gamma}_2/\sqrt{\gamma}_2$ as distributions, then $u = \log(\gamma_1/\gamma_2)$ satisfies the equation*

$$\operatorname{div} \sqrt{\gamma_1\gamma_2}\nabla u = 0.$$

As a consequence, if Ω is C^1, and $\gamma_1 = \gamma_2$ on the boundary, then $\gamma_1 = \gamma_2$.

Proof. Let ϕ be a $C^1(\Omega)$ function, say, which is compactly supported in Ω. Our hypothesis gives

$$0 = \int_\Omega \nabla\sqrt{\gamma_1} \cdot \nabla(\frac{1}{\sqrt{\gamma_1}}\phi) - \nabla\sqrt{\gamma_2} \cdot \nabla(\frac{1}{\sqrt{\gamma_2}}\phi)\, dx$$

If we make the substitution $\phi = \sqrt{\gamma_1}\sqrt{\gamma_2}\psi$, then we have

$$\int_\Omega \sqrt{\gamma_1\gamma_2}\nabla(\log\sqrt{\gamma_1} - \log\sqrt{\gamma_2}) \cdot \nabla\psi\, dx = 0.$$

If $\gamma_1 = \gamma_2$ on the boundary and Ω is C^1, then by Lemma 13.9 we have $\log(\gamma_1/\gamma_2)$ is in $L^{2,1}_0(\Omega)$. We can conclude that this function is zero in Ω from the uniqueness assertion of Theorem 10.25. ∎

Chapter 14

Bessel functions

Chapter 15

Restriction to the sphere

Chapter 16

The uniform Sobolev inequality

In this chapter, we give the proof of a theorem of Kenig, Ruiz and Sogge which can be viewed as giving a generalization of the Sobolev inequality. One version of the Sobolev inequality is that if $1 < p < n/2$, then we have

$$\|u\|_p \leq C(n,p)\|\Delta u\|_p.$$

This can be proven using the result of exercise 8.2 and the Hardy-Littlewood-Sobolev theorem, Theorem 8.10. In our generalization, we will consider more operators, but fewer exponents p. The result is

Theorem 16.1 *Let $L = \Delta + a \cdot \nabla + b$ where $a \in \mathbf{C}^n$ and $b \in \mathbf{C}$ and let p satisfy $1/p - 1/p' = 2/n$. For each f with $f \in L^p$ and $D^2 f \in L^p$ we have*

$$\|f\|_{p'} \leq C\|Lf\|_p.$$

Chapter 17

Inverse problems: potentials in $L^{n/2}$

Chapter 18

Scattering for a two-dimensional system

In this chaper, we begin the study of the scattering theory for a first-order system in two dimensions. The scattering theory for this system is related to the inverse-conductivity problem in two dimensions. We will see that this system is also related to an evolution equation in space called the Davey-Stewartson II system.

The system we study for the next several chapters can be written in the form

$$(D - Q)\psi = 0 \tag{18.1}$$

where D is the first order matrix of differential operators given by

$$D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}.$$

Throughout this chapter, $x = x_1 + ix_2$ will denote a complex variable and we use the standard notation for complex partial derivatives,

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \qquad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$$

The potential Q in (18.1) is a 2×2 off-diagonal matrix,

$$Q = \begin{pmatrix} 0 & q^{12} \\ q^{21} & 0 \end{pmatrix}$$

where the entries q^{ij} are functions on $\mathbf{R}^2$. We will sometimes impose one of the symmetry conditions $Q = Q^*$ or $Q = -Q^*$.

Exercise 18.2 *(a) Find*

$$\bar{\partial}\bar{x}, \quad \bar{\partial}x, \quad \partial\bar{x} \text{ and } \partial x.$$

(b) If f is a function of a complex variable, write the Taylor expansion of f to order 2 using the complex derivatives ∂ and $\bar{\partial}$.

For those who are not familiar with the term scattering theory, we will describe how we understand this term. In the problems we consider, we will compare solutions of a free system (where $Q = 0$) with a system where Q is not zero. We shall see that we can determine Q from the asymptotic behavior of solutions. Scattering theory was developed to attack problems in physics where one tries to determine information at very small scales (for example, the structure of an atom) from measurements made at much larger scales.

18.1 Jost solutions

If $Q = 0$ in (18.1), then we may write down a family of solutions to this system which depend on a complex parameter z

$$\psi_0(x,z) = \begin{pmatrix} e^{ixz} & 0 \\ 0 & e^{-i\bar{x}z} \end{pmatrix}.$$

It is easy for even the most casual observer to see that $D\psi_0 = 0$ where D acts by differentiating in the variables x and $\bar{x}$.

Our goal is to construct solutions of (18.1) which are asymptotic to ψ_0 at infinity. The previous sentence will need to made precise before we can prove theorems. It is reasonable to expect that we can recover Q by studying the ψ and ψ_0. What may be more surprising is that we can recover Q just from the asymptotic behavior of the family $\psi(\cdot, z)$ as z ranges over $\mathbf{C}$. This is one of the goals of the scattering theory we develop in the next several chapters.

The exponential growth of the function ψ_0 is inconvenient. We will find an equation satisfied by $\psi\psi_0^{-1}$ that eliminates the need to deal with functions of exponential growth. We write $\psi(x,z) = m(x,z)\psi_0(x,z)$ and observe that

$$D(m\psi_0) = \begin{pmatrix} (\bar{\partial}m^{11})\psi_0^{11} & (\bar{\partial}m^{12}\psi_0^{22}) \\ \partial(m^{21}\psi_0^{11}) & (\partial m^{22})\psi_0^{22} \end{pmatrix}$$

Thus, we have that

$$\begin{aligned}
D(m\psi_0)\psi_0^{-1} &= Dm^d + \begin{pmatrix} 0 & (\bar\partial(m^{12}e^{-i\bar{x}z}))e^{i\bar{x}z} \\ (\partial(m^{21}e^{ixz}))e^{-ixz} & 0 \end{pmatrix} \\
&= Dm^d + \begin{pmatrix} 0 & (\bar\partial(m^{12}e^{-ix\bar{z}-i\bar{x}z}))e^{ix\bar{z}+i\bar{x}z} \\ (\partial(m^{21}e^{ixz+i\bar{x}\bar{z}}))e^{-ixz-i\bar{x}\bar{z}} & 0 \end{pmatrix} \\
&= Dm^d + (D(m^o A(x,\bar{z})))A(x,-\bar{z})
\end{aligned}$$

The second equality follows since $\bar\partial e^{ix\bar{z}} = 0$. The matrix-valued function A is defined by

$$A(x,z) = A_z(x) = \begin{pmatrix} e^{i(x\bar{z}+\bar{x}z)} & 0 \\ 0 & e^{-ixz-i\bar{x}\bar{z}} \end{pmatrix}.$$

Using the matrix A we may define the operator E_z by

$$E_z f = f^d + f^o A_{\bar{z}} = f^d + A_{-z} f^o.$$

Here, we let A^d denote the diagonal part of a square matrix A and $A^o = A - A^d$ denotes the off-diagonal part of A. Finally, we define an operator D_z by $D_z = E_z^{-1} D E_z$. With this notation, we see that $\psi = m\psi_0$ will solve (18.1) if and only if m satisfies

$$D_z m - Qm = 0. \tag{18.3}$$

18.2 Estimates

To proceed, we will need to fix a norm on matrices and define spaces of matrix valued functions. If A and B are $n \times n$ matrices with complex entries, then we define an inner product by $\langle A, B\rangle = \mathrm{tr}(AB^*)$. Then, the norm is given by

$$|A|^2 = \sum_{j=1}^{n}\sum_{k=1}^{n} |A^{jk}|^2. \tag{18.4}$$

where $A = (A^{jk})$. The norm $|\cdot|$ is often called the Frobenius norm.

Exercise 18.5 *a) If A is a matrix Let $\|A\|_{\mathcal{L}}$ denote the norm of A as an operator on $\mathbf{C}^n$, $\|A\|_{\mathcal{L}} = \sup\{|Ax| : |x| \le 1\}$. Show that these norms satisfy the inequality $|A|_{\mathcal{L}} \le |A|$.*

b) Prove that our matrix norm is mulitiplicative for the matrix product

$$|AB| \le |A||B|.$$

Hint: If C_j denotes the jth column of a matrix C, then use the operator norm to estimate $(AB)_j$. Use part (a) to estimate the operator norm of A in terms of the Frobenius norm.

With this norm, we may define spaces of matrix-valued functions $L^p_\alpha(\mathbf{R}^2) = \{f : \langle y\rangle^\alpha f(y) \in L^p(\mathbf{R}^2)\}$. The norm on this space is

$$\|f\|_{L^p_\alpha(\mathbf{R}^2)} = \left(\int_{\mathbf{R}^2} |f(y)|^p \langle y\rangle^{\alpha p}\, dy\right)^{1/p}.$$

When $p = \infty$, we define $\|f\|_{L^\infty_\alpha(\mathbf{R}^2)} = \operatorname{ess\,sup}\langle y\rangle^\alpha |f(y)|$. Here, we are using $\langle x\rangle = (1 + |x|^2)^{1/2}$. Throughout this section, we will let p denote an exponent in the open interval $(1, 2)$ and then $\tilde{p}$ is defined by the relation $1/\tilde{p} = 1/p - 1/2$. We begin with the following simple extension of Hölder's inequality.

Proposition 18.6 *Suppose that p, q and r lie in $[1, \infty]$ and $1/p = 1/q + 1/r$, that α and β lie in $\mathbf{R}$. If $f \in L^q_\alpha(\mathbf{R}^2)$ and $g \in L^r_\beta(\mathbf{R}^2)$, then $fg \in L^p_{\alpha+\beta}(\mathbf{R}^2)$ and*

$$\|fg\|_{L^p_{\alpha+\beta}(\mathbf{R}^2)} \le \|f\|_{L^q_\alpha(\mathbf{R}^2)} \|g\|_{L^r_\beta(\mathbf{R}^2)}.$$

Proof. The proof is a simple application of Hölder's inequality. ∎

We now consider the equation

$$Du = f.$$

One solution of this equation is given by the operator G defined by

$$G(f)(x) = \frac{1}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} x-y & 0 \\ 0 & \bar{x}-\bar{y} \end{pmatrix}^{-1} f(y)\, dy = \begin{pmatrix} g(f)(x) & 0 \\ 0 & \bar{g}(f)(x) \end{pmatrix}$$

where g and $\bar{g}$ are the Cauchy transform and the corresponding operator for ∂. (See exercise 8.3) We summarize some well-known properties of this operator G.

Proposition 18.7 *Fix p with $1 < p < 2$. If $f \in L^p(\mathbf{R}^2)$, then we have*

$$\|G(f)\|_{L^{\tilde{p}}(\mathbf{R}^2)} + \|\nabla G(f)\|_{L^p(\mathbf{R}^2)} \le C\|f\|_{L^p(\mathbf{R}^2)}.$$

The constant C depends only on p and $\tilde{p}$ is defined by $1/\tilde{p} = 1/p - 1/2$.

The function $u = G(f)$ is the unique solution of $Du = f$ which lies in the space $L^{\tilde{p}}(\mathbf{R}^2)$.

Next we define an operator G_z by $G_z = E_z^{-1} G E_z$. The next Corollary follow easily from Proposition 18.7.

Corollary 18.8 *If $1 < p < 2$ and f is in $L^p(\mathbf{R}^2)$, then $u = G_z(f)$ satisfies $D_z u = f$. We have*

$$\|G_z(f)\|_{L^{\tilde{p}}(\mathbf{R}^2)} \le C\|f\|_{L^p(\mathbf{R}^2)}$$

and for each f in L^p, the map $z \to G_z(f)$ from $\mathbf{C}$ into $L^{\tilde{p}}$ is continuous or to put it another way, the map $z \to G_z$ continuous from $\mathbf{C}$ into $\mathcal{L}(L^p, L^{\tilde{p}})$ with the strong operator topology.

Proof. It is easy to see that $G_z(f)$ solves $D_z u = f$. As the map E_z is an invertible norm preserving map on L^p, the norm estimate for G_z follows from the corresponding result for G in Proposition 18.7. To establish the strong convergence of the family of operators G_z, we observe that

$$|G_z(f)(x)| \le I_1(|f|)(x)$$

where I_1 is the Riesz potential defined in (8.4). For x and y fixed, we have the

$$\lim_{w\to z} A(x-y, w)f(y) = A(x-y, z)f(y).$$

If $I_1(|f|)(x)$ is finite, then $|f(y)|/|x-y|$ is in $L^1(\mathbf{R}^2)$ and the dominated convergence theorem implies

$$\lim_{w\to z} G_w(f) = G_z(f).$$

Another application of the dominated convergence theorem implies that

$$\lim_{w\to z} \|G_w(f) - G_z(f)\|_{L^p(\mathbf{R}^2)} = 0.$$

∎

We will find solutions to the equation (18.3) in the form $(I - G_z Q)^{-1}(I_2)$ where I_2 is the 2×2 identity matrix. To proceed, we will need to show that the inverse $(I - G_z Q)^{-1}$ exists and is differentiable. We begin by establishing the invertibility.

Proposition 18.9 *Let $k \ge 0$ and assume that $Q \in L^2_k(\mathbf{R}^2)$. If $2 < \tilde{p} < \infty$, we have that*

$$G_z Q : L^{\tilde{p}}_{-k} \to L^{\tilde{p}}_{-k}$$

and we have the estimate

$$\|G_z Q(f)\|_{L^{\tilde{p}}_{-k}(\mathbf{R}^2)} \le C\|Q\|_{L^2_k(\mathbf{R}^2)}\|f\|_{L^{\tilde{p}}_{-k}(\mathbf{R}^2)}.$$

The constant C depends only on p.

Proof. From Proposition 18.6, we have

$$\|Qf\|_{L^p(\mathbf{R}^2)} \le \|Q\|_{L^2_k(\mathbf{R}^2)}\|\tilde{f}\|_{L^{\tilde{p}}_{-k}(\mathbf{R}^2)}.$$

From Proposition 18.8, we have $\|G_z(Qf)\|_{L^{\tilde{p}}(\mathbf{R}^2)} \le C\|Qf\|_{L^p(\mathbf{R}^2)}$. Finally, we have $L^{\tilde{p}}_{-k} \subset L^{\tilde{p}}$ since $\langle y\rangle \ge 1$ and the estimate of the Proposition follows. ∎

Remark. When $p = 4/3$, it is known that $C(p) \le 1$. See Lieb and ...

The following Lemma shows that the map $z \to G_z Q$ is continuous into operators on $L^{\tilde p}_{-k}$ in the operator norm. Note that this is a stronger result than we proved in Corollary 18.9. In a future version of these notes, Corollary 18.9 will be disappeared.

Lemma 18.10 *Let $1 < p < 2$ and suppose that $k \ge 0$. If $Q \in L^{\tilde p}_{-k}$, then the map $z \to G_z Q$ is continuous as a map from the complex plane into $\mathcal{L}(L^{\tilde p}_{-k})$. If we assume that $(I - G_{z_0} Q)$ is invertible, then the inverse exists in a neighborhood of z_0 and the map $z \to (I - G_z Q)^{-1}$ is continuous at z_0.*

Proof. We begin by showing that the map $z \to G_z Q$ is continuous. To see this, fix $\epsilon > 0$. We first observe that we may find Q_0 which compactly supported and so that $\|Q_0 - Q\|_{L^2_k} < \epsilon$. Then, Hölder's inequality, Theorem 18.6, and the Hardy-Littlewood-Sobolev inequality, Theorem 8.10, imply that

$$\|G_z(Q - Q_0)f\|_{L^{\tilde p}_{-k}} \le C\|Q - Q_0\|_{L^2_k}\|f\|_{L^{\tilde p}_{-k}}$$

as long as $k \ge 0$.

Since $\operatorname{supp} Q_0$ is compact, we may find $R > 0$ so that $\operatorname{supp} Q_0 \subset B_R(0)$. We can find a constant C so that

$$|G_z Q_0 f(x)| \le \frac{C\|Q_0\|_{L^2_k}\|f\|_{L^{\tilde p}_{-k}}}{|x|}, \qquad |x| > R.$$

Recalling that $\tilde p > 2$ and $k \ge 0$, it follows from the above that we may choose R_1 so that

$$\|G_z Q_0 f\|_{L^{\tilde p}_{-k}(\{x:|x|>R_1\})} \le \epsilon\|f\|_{L^{\tilde p}_{-k}}.$$

Finally, we observe that our matrix valued function $A(x,z)$ satisfies $|A(x,z) - A(x,w)| \le C|x||z-w|$. This and the compact support of Q_0 allow us to conclude that we have a constant C so that

$$\begin{aligned}\|G_z Q_0 f - G_w Q_0 f\|_{L^{\tilde p}_{-k}} &\le \|(E_z^{-1} - E_w^{-1})G E_z Q_0 f\|_{L^{\tilde p}_{-k}(\{x:|x|<R_1\})} + \|E_w^{-1} G(E_z - E_w)Q_0 f\|_{L^{\tilde p}_{-k}} \\ &\le C(R + R_1)|z-w|\|f\|_{L^{\tilde p}_{-k}}.\end{aligned}$$

From the above observations, we see that we have

$$\begin{aligned}\|G_z Q f - G_w Q f\|_{L^{\tilde p}_{-k}} &\le \|G_z(Q - Q_0)f\|_{L^{\tilde p}_{-k}} + \|G_w(Q - Q_0)f\|_{L^{\tilde p}_{-k}} \\ &\quad + \|(G_z - G_w)(Q_0)f\|_{L^{\tilde p}_{-k}(\{x:|x|<R\})} \\ &\quad + \|(G_z)(Q_0)f\|_{L^{\tilde p}_{-k}(\{x:|x|\ge R\})} + \|(G_w)(Q_0)f\|_{L^{\tilde p}_{-k}(\{x:|x|\ge R\})} \\ &\le C\epsilon\|f\|_{L^{\tilde p}_{-k}} + C(R + R_1)|z-w|\|f\|_{L^{\tilde p}_{-k}}.\end{aligned}$$

If we require $|z-w|$ to be small so that $C(R+R_1)|z-w| < \epsilon$, we obtain that the operator norm of $G_zQ - G_wQ$ is at most a multiple of ϵ. We have established the norm continuity of G_zQ.

To establish the continuity of the inverse, we write

$$(I-G_zQ)^{-1} = (I-G_{z_0}Q+G_{z_0}Q-G_zQ)^{-1} = (I-G_{z_0}Q)^{-1}\sum_{j=0}^{\infty}((I-G_{z_0}Q)(G_{z_0}Q-G_zQ))^j.$$

The norm continuity of the map $z \to G_zQ$ now implies that the inverse map is also continuous. ∎

Corollary 18.11 *Let $1 < p < 2$ and suppose that $k \geq 0$. Let $C = C(p)$ be the constant in Proposition 18.9. If we have $C\|Q\|_{L^2_k(\mathbf{R}^2)} < 1$, then the operator $(I-G_zQ)$ is invertible on $L^{\tilde{p}}_{-k}$ and the norm of the inverse operator satisfies*

$$\|(I-G_zQ)^{-1}\|_{\mathcal{L}(L^{\tilde{p}}_{-k})} \leq \frac{1}{1-C\|Q\|_{L^2_k}}.$$

In addition, the map $z \to (I-G_zQ)^{-1}$ is continuous as a map into the operators on $L^{\tilde{p}}_{-k}$ with the norm topology.

Proof. We write

$$(I-G_zQ)^{-1} = \sum_{k=0}^{\infty}(G_zQ)^k.$$

We may use Proposition 18.9 to estimate each term in this series and sum to obtain the estimate of the Proposition.

To obtain the continuity of the inverse map, we write

$$((I-G_wQ)^{-1} - (I-G_zQ)^{-1})f = ((I-G_wQ)^{-1}(G_zQ-G_wQ)(I-G_zQ)^{-1})f.$$

Since the norm of $(I-G_wQ)^{-1}$ is bounded in w, it is clear that the strong continuity of the map $z \to G_zQ$ implies the continuity of the inverse map (see the proof of Lemma A.5). ∎

Corollary 18.12 *Suppose that $1 < p < 2$ and that $k > 2/\tilde{p}$ and $Q \in L^2_k$ with $C\|Q\|_{L^2_k}$. We may find m a solution of (18.3) with $\|m(\cdot,z\|_{L^{\tilde{p}}_-k} \leq C$ and m is the only solution in $L^{\tilde{p}}_{-k}$. Furthermore, $m(\cdot,z) - I$ lies in $L^{\tilde{p}}$ and there is only one solution of (18.3) with $m(\cdot,z) - I_2$ in $L^{\tilde{p}}$.*

Proof. Our condition on $\tilde{p}$ and k guarantees that I_2 is in $L^{\tilde{p}}_{-k}$. Set $m(\cdot,z) = (I - G_zQ)^{-1}(I_2)$. ∎

18.3 Notes

The material we discuss in this chapter is presented formally in Fokas [11], Fokas and Ablowitz [12], some proofs are sketched in Beals and Coifman [4], and the results are worked out in complete detail in Sung [32, 33, 34].

Chapter 19

Global existence of Jost solutions

Our next goal is to show that if Q satisfies $Q = Q^*$, then we have the existence of the Jost solutions m without any size restriction on Q and we have the estimate

$$\sup_{z\in\mathbf{C}} \|m(\cdot, z)\|_{L^{\tilde{p}}_{-k}} \leq C(Q).$$

We will assume some smoothness on Q to do this. Our goal is to develop a complete theory when the potential is in the Schwartz class.

19.1 Uniqueness

In this section, we establish uniqueness of solutions to the equation (18.3), $(D_z - Q)m = 0$. We begin by studying a scalar equation which contains all of the analytic difficulties. Under the assumption that $Q = Q^*$, we are able to reduce the system to several scalar equations.

Lemma 19.1 *Suppose that f lies in $L^p(\mathbf{R}^2) \cap L^{p'}(\mathbf{R}^2)$ for some p with $1 < p < 2$, then the Cauchy transform of f, $g(f)$ is continuous and satisfies*

$$\|g(f)\|_\infty \leq C(\|f\|_p \|f\|_{p'})^{1/2}.$$

Proof. We let $R > 0$ and write

$$g(f)(x) = \frac{1}{\pi}\left(\int_{|x-y|<R} \frac{f(y)}{x-y}\,dy + \int_{|x-y|>R} \frac{f(y)}{x-y}\,dy\right) = \frac{1}{\pi}(I(x) + II(x)).$$

From Hölder's inequality, we have

$$|I(x)| \leq \|f\|_{p'} \left(\int_{B_R(0)} |y|^{-p}\, dy \right)^{1/p} = \|f\|_{p'} \left(\frac{2\pi}{2-p} \right)^{1/p} R^{\frac{2}{p}-1}.$$

Another application of Hölder's inequality gives that

$$II(x) \leq \|f\|_p \left(2\pi \int_R^\infty r^{1-p'}\, dr \right)^{1/p'} = \|f\|_p \left(\frac{2\pi}{p'-2} \right)^{1/p'} R^{1-\frac{2}{p}}.$$

Thus, $g(f)(x)$ satisfies

$$|g(f)(x)| \leq \frac{1}{\pi} \left(\|f\|_p \left(\frac{2\pi}{p'-2} \right)^{1/p'} R^{1-\frac{2}{p}} + \|f\|_{p'} \left(\frac{2\pi}{2-p} \right)^{1/p} R^{\frac{2}{p}-1} \right).$$

The minimum on the right-hand side of this expression will occur if $R^{-1+2/p}$ a multiple of $(\|f\|_p/\|f\|_{p'})^{1/2}$. Thus, we may find a constant $C(p)$ so that

$$\|g(f)\|_\infty \leq C \|f\|_p^{1/2} \|f\|_{p'}^{1/2}. \tag{19.2}$$

To see that $g(f)$ is continuous, we fix f in $L^p \cap L^{p'}$ and approximate f by a sequence of smooth functions which converge in L^p and is bounded in $L^{p'}$. (Unless $p = 1$, it is easy to arrange convergence in L^p and $L^{p'}$.) The estimate (19.2) implies that $g(f_j)$ converges uniformly to $g(f)$ and hence $g(f)$ is continuous. ∎

Exercise 19.3 *Show that we have*

$$\|g(f)\|_{C^\alpha} \leq C(\alpha, p)\|f\|_p$$

if $\alpha = 1 - 2/p$, $2 < p < \infty$. *Here, we are using* $\|\cdot\|_{C^\alpha}$ *to denote the* C^α *semi-norm*

$$\|f\|_{C^\alpha} = \sup\{x \neq y : |f(x) - f(y)|/(|x-y|^\alpha)\}.$$

In the next theorem, we use the space $C_0(\mathbf{R}^2)$. This the closure in the uniform norm of the continuous functions with compact support.

Theorem 19.4 *(Vekua [39]) Suppose that* q_1 *and* q_2 *lie in* $L^p \cap L^{p'}$ *for some* p *with* $1 < p < 2$. *If* f *is in* L^r *for some* r *with* $p \leq r < \infty$ *or* f *is in* $C_0(\mathbf{R}^2)$ *and satisfies the pseudo-analytic equation*

$$\bar\partial f - q_1 f - q_2 \bar f = 0,$$

then $f = 0$.

Proof. We set

$$q(x) = \begin{cases} q_1(x), & \text{if } f(x) = 0 \\ q_1(x) + \frac{f(x)}{\bar{f}(x)} q_2(x), & \text{if } f(x) \neq 0. \end{cases}$$

With this definition, we have that $\bar{\partial} f - qf = 0$ and we have $q \in L^p \cap L^{p'}$ for $1 < p < 2$. Our hypothesis on q and Lemma 19.1 imply that $u = C(q)$ is bounded. The pseudo-analytic equation implies that $g = fe^{-u}$ is analytic. Since u is bounded, we have that g lies in $L^{p'}$ (or C_0) whenever f does. Thus, Liouville's theorem implies that g must be identically 0. ∎

Exercise 19.5 *(Nachman) Extend Theorem 19.4 to the case when $q_j \in L^2$ for $j = 1, 2$.*

We now show how to use the above the result of Vekua to establish uniqueness to our first-order system, (18.3).

Theorem 19.6 *Suppose that $Q = Q^*$ and Q lies in $L^p \cap L^{p'}$, for some p with $1 < p < 2$. If $D_z m - Qm = 0$ and m lies in L^r for some r with $\infty > r \geq p$, then $m = 0$.*

Proof. Fix z. We define four functions, $u^\pm$ and $v^\pm$ by

$$\begin{aligned} u^\pm(x) &= m^{11}(x,z) \pm \exp(-ixz - i\bar{x}\bar{z})\bar{m}^{21}(x,z) \\ v^\pm(x) &= \exp(ix\bar{z} + i\bar{x}z)\bar{m}^{12}(x,z) \pm m^{22}(x,z). \end{aligned}$$

Each of these functions will be a solution of the equation $\bar{\partial} f - qf = 0$. To see this for $u^\pm$, we start with the equations

$$\begin{aligned} \bar{\partial} m^{11}(x,z) &= q^{12}(x) m^{21}(x,z) \\ \bar{\partial}(\bar{m}^{21}(x,z)\exp(-ixz - i\bar{x}\bar{z}) &= \bar{q}^{21}(x)\bar{m}^{11}(x,z)\exp(-ixz - i\bar{x}\bar{z}). \end{aligned}$$

Adding these expressions and using that $Q^* = Q$, we obtain that

$$\begin{aligned} &\bar{\partial} m^{11}(x,z) \pm \bar{m}^{21}(x,z)\exp(-ixz - i\bar{x}\bar{z}) \\ &= q^{12}(x) m^{21}(x,z) \pm q^{12}(x)\bar{m}^{11}(x,z)\exp(ixz + i\bar{x}\bar{z}) \pm \bar{m}^{11} \\ &= \pm q^{12}\exp(-ixz - i\bar{x}\bar{z})\bar{u}^\pm. \end{aligned}$$

A similar calculation gives that

$$\partial v^\pm(x) = q^{21}(x)\exp(ixz + i\bar{x}\bar{z})\bar{v}^\pm(x).$$

These equations, and our hypotheses on Q and m allows us to use Theorem 19.4 to show that $u^\pm = v^\pm = 0$. It follows from these four equations that the four components of m are zero. ∎ Check this.

Next, we observe that the map $f \to G_z Qg$ is a compact operator.

Theorem 19.7 *Let $1 < p < 2$ and suppose that $Q \in L^2_k$. Then the operator $f \to G_z Q f$ is a compact operator on $L^{\tilde{p}}_{-k}$.*

Proof. We first consider the case where Q is bounded and compactly supported. If Q is supported in a ball B, then we have that $GE_z^{-1}Qf$ will have analytic or anti-analytic components outside of the support of Q. By examining the Laurent expansions of these functions, we may see that $f \to G_z Q f$ is a compact map into $L^{\tilde{p}}_{-k}(\mathbf{C} \setminus B)$. On the set B, we may use that $GE_z^{-1}Qf$ is in the Sobolev space $L^{q,1}(B)$ for $q < \infty$ and the Rellich compactness theorem to conclude that $G_z Q$ is a compact map into $L^{\tilde{p}}_{-k}(B)$. Thus we have proven our theorem in the special case that Q is bounded and compactly supported.

If Q is an arbitrary element in L^2_k, then we may approximate Q in the space L^2_k by a sequence $\|Q_j\}$ where each Q_j is compactly supported and bounded. We have that the sequence of operators $G_z Q_j$ converges in operator norm to the operator $G_z Q$. As the set of compact operators is closed in the operator topology, we may conclude that the operator $G_z Q$ is compact.

∎

19.2 Existence of solutions.

In this section, we establish the existence of solutions to an integral equation. This result is the main step in establishing existence of solutions to the equation (18.3).

Theorem 19.8 *Fix p with $1 < p < 2$. Suppose that $Q \in L^r \cap L^{r'}$, $Q = Q^*$ and $Q \in L^2_k$ then for each h in $L^{\tilde{p}}_{-k}$, the integral equation*

$$(I - G_z Q)(f) = h \tag{19.9}$$

has a unique solution in $L^{\tilde{p}}_{-k}$. If we set $f(x,z) = (I - G_z Q)^{-1}(h)$ is continuous as a map from $\mathbf{C}$ into $L^{\tilde{p}}_{-k}$.

Proof. We begin by showing that solutions of (19.9) are unique. Suppose that $(I - G_z Q)(f) = 0$ and f is in $L^{\tilde{p}}_{-k}$. Since we assume that $Q \in L^2_k$, if follows from Hölder's inequality that Qf is in L^p we conclude that $f = G_z(Qf)$ is in $L^{\tilde{p}}$ by Theorem 8.10. From this, we conclude that $D_z f - Qf = 0$. It follows from Theorem 19.4 that $f = 0$.

To show existence, we begin by observing that Proposition 18.6 and Theorem 8.10 imply that $(I - G_z Q)$ is a bounded operator on $L^{\tilde{p}}_{-k}$. Also, from Theorem 19.7, we have that the operator $f \to G_z Q f$ is compact. Since we have shown that the operator $f \to (I - G_z Q)f$ is injective, it follows from the Fredholm theorem that $I - G_z Q$ is invertible.

The continuity follows from Lemma 18.10. ∎

Corollary 19.10 *Suppose that $1 < p < 2$ and $k\tilde{p} > 2$ and let $Q \in L^2_k$. If Q is small in L^2_k or $Q = Q^*$ and $Q \in L^r \cap L^{r'}$ for some r with $1 < r < 2$, then we may find a unique solution of (18.3) with $m(\cdot, z) - I_2 \in L^{\tilde{p}}$.*

19.3 Behavior for large z

Our next theorem shows that m is bounded for large z and in fact we have

$$\lim_{|z|\to\infty} \|I_2 - m(\cdot, z)\|_{L^{\tilde{p}}_{-k}} = 0.$$

We begin by establishing an integral equation for the diagonal part of m, m^d.

Lemma 19.11 *Suppose that $Q \in L^2_k$ with $k \geq 0$ and h is in $L^{\tilde{p}}_{-k}$ with $h^o = 0$. A matrix-valued function m in $L^{\tilde{p}}_{-k}$ satisfies the integral equation $(I - G_z Q)m = h$ if and only if the diagonal and off-diagonal parts satisfy*

$$m^d(\cdot, z) - GQG_z Q m^d(\cdot, z) = h \tag{19.12}$$
$$m^o(\cdot, z) = G_z Q m^d(\cdot, z) \tag{19.13}$$

Proof. If we have $m - G_z Q m = h$, we may separate this equation into the diagonal and off-diagonal parts of the matrix and obtain the equations

$$m^d - GQm^o = h \tag{19.14}$$
$$m^o - G_z Q m^d = 0 \tag{19.15}$$

If we substitute the second equation into the first, we obtain (19.12) and of course the second equation is (19.13).

Now suppose that we have the equation (19.12). If we define $m^o = G_z Q m^d$, we immediately obtain the integral equation (19.9). ∎

Theorem 19.16 *Fix p with $1 < p < 2$ and $k \geq 0$. Let $Q \in L^2_k$, fix p. If $\nabla Q \in L^2$ and $Q \in L^\infty$, then we have the estimate*

$$\|GQG_zQf\|_{L^{\tilde{p}}} \leq C(Q)|z|^{-1}\|f\|_{L^{\tilde{p}}_{-k}}.$$

Proof. We give a detailed proof for one component of GQG_zQf. The other components may be handled similarly. We begin by writing T as the operator which takes f^{11} to $(GQG_zQf)^{11}$,

$$\begin{aligned} Tf^{11}(x_1) &= (GQG_zQf)^{11}(x_1) \\ &= \frac{1}{\pi^2}\int_{\mathbf{C}} \frac{q^{12}(x_2)a^2(x_2-x_3,-z)q^{21}(x_3)}{(x_1-x_2)(\bar{x}_2-\bar{x}_3)} f^{11}(x_3)\,dx_2\,dx_3. \end{aligned}$$

We write $a^2(x_2-x_3,z) = \frac{1}{iz}\partial_{x_2}a^2(x_2-x_3,z)$, substitute this into the previous displayed equation and integrate by parts to obtain

$$\begin{aligned} Tf^{11}(x_1) &= \frac{1}{iz\pi^2}\int_{\mathbf{C}^2} \frac{1}{x_1-x_2} q^{12}(x_2)q^{21}(x_2)f^{11}(x_2)\,dx_2 \\ & \quad \frac{1}{iz\pi^2}\int_{\mathbf{C}^2} \frac{\partial q^{12}(x_2)a^2(x_2-x_3,z)q^{21}(x_3)}{(x_1-x_2)(\bar{x}_2-\bar{x}_3)} f^{11}(x_3)\,dx_2\,dx_3 \\ & \quad \frac{1}{iz\pi^2}\text{p.v.}\int_{\mathbf{C}} \frac{1}{(x_1-x_2)^2} q^{12}(x_2)\int \frac{a^2(x_2-x_3,z)q^{21}(x_3)f^{11}(x_3)}{\bar{x}_2-\bar{x}_3}\,dx_3dx_2 \\ &= \frac{1}{iz\pi^2}(I+II+III). \end{aligned}$$

We estimate these three terms.

By Theorem 8.10 and Hölder's inequality, we have

$$\|I\|_{\tilde{p}} \leq \|Q\|_\infty \|Q\|_{L^2_k}\|f\|_{L^{\tilde{p}}_{-k}}.$$

For the second term, II, we use Hölder and Theorem 8.10 again to obtain

$$\|II\|_{L^{\tilde{p}}} \leq C\|DQ\|_{L^2}\|Q\|_{L^2_k}$$

For III, we use the Calderón-Zygmund theorem, Theorem 6.10, to obtain that

$$\|III\|_{L^{\tilde{p}}} \leq C\|Q\|_\infty\|Q\|_{L^2_k}\|f\|_{L^{\tilde{p}}_{-k}}.$$

These estimates complete the proof of the theorem. ∎

Finally, we can give the main result of this chapter.

Theorem 19.17 *Let $\tilde{p}k > 2$. If Q is in $\mathcal{S}(\mathbf{R}^2)$, then $m(\cdot, z) = (I - G_zQ)^{-1}(I_2)$ satisfies*

$$\|m(\cdot, z)\|_{L^{\tilde{p}}_{-k}} \leq C.$$

In addition, m is a solution of (18.3) and for each z, $m(\cdot, z)$ is the only solution of this equation with the condition that $m(\cdot, z) - I_2$ is in $L^{\tilde{p}}$.

Proof. We solve the iterated integral equation (19.12). For z large, Theorem 19.16 tells us that we can solve the integral equation (19.12) with a Neumann series. Thus, we may find R so that $\|m(\cdot, z)\|_{L^{\tilde{p}}_{-k}} \leq C$ if $|z| \geq R$. For z less then R, Proposition 18.10 tells us that $z \to \|m(\cdot, z)\|_{L^{\tilde{p}}_{-k}}$ is continuous function and hence this function is bounded on the compact set $\{z : |z| \leq R\}$.

If m and m' are two solutions of (18.3) with $m - I_2$ in $L^{\tilde{p}}$, then $m - m'$ is a solution of $(D_z - Q)(m - m') = 0$ with $m - m' \in L^{\tilde{p}}$. Theorem 19.4 implies that $m = m'$. ∎

Remark. One puzzling feature of the above Theorem is that we know that $\|m(\cdot, z)\|_{L^{\tilde{p}}_{-k}}$ is bounded, but we do not have a quantitative estimate for this bound in terms of Q. Can you provide such an estimate?

Chapter 20

Differentiability of the Jost solutions

In this section, we discuss the smoothness of the Jost solutions m to (18.3). We shall see that decay of Q leads to smoothness of m in the z variable and smoothness in Q leads to smoothness of m in the x variable.

When we establish the differentiability of m with respect to z, we will also establish remarkable property of the solutions m. These solutions satisfy an equation in the variable z which is of the same general form as the equation (18.3). This equation will be called the $\partial/\partial\bar{z}$-equation In this equation, the function which takes the place of the potential Q will be called the scattering data S. This scattering data will also appear in an asymptotic expansion of the solution m. If we recall the solutions ψ_0 introduced at the beginning of Chapter 18, we see that these functions are holomorphic with respect to the parameter z. When the potential Q is not zero, then the scattering data tells us how far the solutions m are from being holomorphic.

From these results, we will see that if Q is in the Schwartz function, then the corresponding scattering data is also in $\mathcal{S}(\mathbf{R}^2)$. This will be used when we study how to recover the potential Q from the scattering map.

Throughout this chapter we continue to let p be an exponent in the range $1 < p < 2$.

20.1 Differentiability of the Jost solution with respect to x.

We begin with the equation (18.3) for m. As in Lemma 19.11, we may iterate this equation. It turns out that the off-diagonal part of m is oscillatory. The proper function to study is $n(x, z) = E_z m(x, z)$. It is is easy to see that the function n will satisfy the following iterated integral equation.

$$n - GQ_zGQ_zn = I_2 \tag{20.1}$$

In this equation, we introduce the notation Q_z to stand for the operator given by $Q_zh = E_zQE_z^{-1}h$. We begin with a Lemma on the regularity of solutions of the equation (20.1).

Lemma 20.2 *Fix p with $1 < p < 2$ and let Q be in L^2_k for some $k \geq 0$ and n be in $L^{\tilde{p}}_{-k}$ and satisfy the integral equation*

$$n - GQ_zGQ_zn = f.$$

If ∇f lies in L^p, then ∇n lies in L^p and we have the estimate

$$\|\nabla n\|_{L^p} \leq C\|Q\|^2_{L^2_k}\|n\|_{L^{\tilde{p}}_{-k}} + \|\nabla f\|_{L^p}.$$

Proof. We have the representation for n

$$n = GQ_zGQ_zn + f.$$

Our assumptions imply that Q_zn lies in L^p with respect to the x-variable. Then Hardy-Littlewood-Sobolev and Hölder's inequality imply Q_zGQ_zn lies in L^p. Applying the Hardy-Littlewood-Sobolev and the Calderón-Zygmund theorem gives that n lies in $L^{\tilde{p}}$ and that ∇n lies in L^p with respect to the x-variable. ∎

Theorem 20.3 *Let $m \in L^{\tilde{p}}_{-k}$ be a solution of $(I - G_zQ)m = h$ and set $n = E_zm$. Suppose that $\partial^\alpha Q/\partial x^\alpha$ lies in L^2_k for all α with $|\alpha| \leq \ell$ and that h is a diagonal-matrix valued function with $\partial^\alpha h/\partial x^\alpha$ in L^p for all multi-indices α with $1 \leq |\alpha| \leq \ell$. Then $\nabla\partial^\alpha n^d/\partial x^\alpha$ lies in L^p for all α with $|\alpha| \leq \ell$.*

Proof. We establish the result for one component and leave the details for the other component to the reader. We write

$$n^{11}(x,z) = \frac{1}{\pi^2}\int_{\mathbf{R}^4} \frac{q^{12}(x_1)a^2(x_2 - x_1, z)q^{21}(x_2)n^{11}(x_2,z)}{(x - x_1)(\bar{x}_1 - \bar{x}_2)}\,dx_1\,dx_2 + h^{11}(x).$$

We make the change of variables $x_1 = x - w_1$ and $w_2 = x_1 - x_2$ to obtain

$$n^{11}(x,z) - \frac{1}{\pi^2}\int_{\mathbf{R}^4} \frac{q^{12}(x - w_1)a^2(w_2, -z)q^{21}(x - w_1 - w_2)n^{11}(x - w_1 - w_2, z)}{w_1\bar{w}_2}\,dw_1\,dw_2 = h^{11}(x). \tag{20.4}$$

We will use induction to show that n has ℓ derivatives.

According to Lemma 20.2, n has one derivative. Let α be a multi-index of length j with $1 \leq j \leq \ell$ and assume the induction hypothesis that n^d has j derivatives. If we differentiate with (20.4) with respect to x, we obtain

$$\begin{aligned} &\frac{\partial^\alpha}{\partial x^\alpha} n^{11}(x,z) \\ &-\frac{1}{\pi^2}\int_{\mathbf{R}^4} \frac{q^{12}(x-w_1)a^2(w_2,-z)q^{21}(x-w_1-w_2)\frac{\partial^\alpha}{\partial x^\alpha}n^{11}(x-w_1-w_2,z)}{w_1\bar{w}_2}\,dw_1\,dw_2 \\ &= \frac{\partial^\alpha}{\partial x^\alpha}h^{11}(x) \\ &\quad +\frac{1}{\pi^2}\sum_{\beta+\gamma+\delta=\alpha,\delta\neq\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}\int_{\mathbf{R}^4}\frac{a^2(w_2,-z)}{w_1\bar{w}_2}\frac{\partial^\beta}{\partial x^\beta}q^{12}(x-w_1) \\ &\quad \times\frac{\partial^\gamma}{\partial x^\gamma}q^{21}(x-w_1-w_2)\frac{\partial^\delta}{\partial x^\delta}n^{11}(x-w_1-w_2,z)\,dw_1\,dw_2. \end{aligned} \tag{20.5}$$

Changing variables again, we may rewrite the right-hand side as terms of the form

$$g(a^2(\cdot,-z)(\frac{\partial^\beta q^{12}}{\partial x^\beta})a^2(\cdot,z)(\bar{g}(\frac{\partial^\gamma q^{21}}{\partial x^\gamma}\frac{\partial^\delta n^{11}}{\partial x^\delta}))(\cdot,z)$$

where g is our new notation for the Cauchy transform. Given our induction hypothesis and the hypotheses on n, we have the right-hand side of (20.5) has one derivative in L^p. Hence, we conclude from Lemma 20.2 that $\nabla\partial^\alpha n^{11}/\partial x^\alpha$ lies in L^p, also. ∎

20.2 Differentiability with respect to z

We discuss the differentiability of the operator G_zQ with respect to z. These results will be used to differentiate the solution $m(x,z)$ of (18.3) with respect to z.

Proposition 20.6 *If $f \in L^1_{\ell-1}(\mathbf{R}^2)$, then for $j = 1, \cdots, \ell$ the functions*

$$\frac{\partial^{\ell-j}}{\partial \bar{z}^{\ell-j}}\frac{\partial^j}{\partial z^j}G_z(f)(x)$$

are continuous for $(x,z) \in \mathbf{R}^4$ and

$$\sup_{x,z}\left|\langle x\rangle^{1-\ell}\frac{\partial^{\ell-j}}{\partial \bar{z}^{\ell-j}}\frac{\partial^j}{\partial z^j}G_z(f)(x)\right| \leq C(\ell)\|f\|_{L^1_{\ell-1}(\mathbf{R}^2)}.$$

Proof. As $G_z(f)^d$ is independent of z, we only need to consider $(G_z f)^o$. To simplify the notation, we only consider one component. The other component may be handled similarly. Differentiating under the integral sign gives

$$\frac{\partial^{\ell-j}}{\partial \bar{z}^{\ell-j}} \frac{\partial^j}{\partial z^j} G_z f(x)^{12} = \frac{i}{\pi} \int (i(x-y))^{\ell-j-1} (i(\bar{x}-\bar{y}))^j e^{i(\bar{z}(x-y)+z(\bar{x}-\bar{y}))} f^{12}(y)\, dy.$$

It is easy to see that the right-hand side is bounded by a multiple of

$$\langle x \rangle^{\ell-1} \int_{\mathbf{R}^2} |f^{12}(y)| \langle y \rangle^{\ell-1}\, dy.$$

∎

Exercise 20.7 *Use the Hausdorff-Young inequality to show that if f is in $L^p(\mathbf{R}^2)$, for some p with $1 \leq p \leq 2$, and*

$$m(x,z) = \frac{\partial}{\partial \bar{z}} G_z(f)(x)$$

then we have a constant C so that

$$\int \sup_x |m(x,z)|^{p'}\, dz \leq C \|f\|_{L^p}^{p'}.$$

Exercise 20.8 *Suppose that $f \in L^1(\mathbf{R}^2)$. Show that*

$$\lim_{|z| \to \infty} \frac{\partial}{\partial \bar{z}} G_z(f)(x) = 0$$

and the convergence is uniform in x.

Corollary 20.9 *Let p and q lie in $(1,\infty)$. Suppose that $k > 2/p'$ and $j > 2/q$. Then the map*

$$z \to G_z$$

is a strongly differentiable map from the plane into $\mathcal{L}(L^p_k(\mathbf{R}^2), L^q_{-j}(\mathbf{R}^2))$.

Proof. Our condition on p and k implies that $L^p_k(\mathbf{R}^2) \subset L^1(\mathbf{R}^2)$. Thus we may use Proposition 20.6, to see that the map $z \to G_z f(x)$ is differentiable in the point-wise sense and the derivative is bounded. Since we have $L^\infty(\mathbf{R}^2) \subset L^q_{-j}$, the dominated convergence theorem implies that

$$\lim_{h \to 0} \frac{1}{|h|} \|G_{z+h} f - G_z f - h \frac{\partial}{\partial z} G_z(f) - \bar{h} \frac{\partial}{\partial \bar{z}} G_z(f)\|_{L^q_{-j}(\mathbf{R}^2)} = 0.$$

∎

Corollary 20.10 *Let $k \geq 0$, and $1 < p < 2$. Suppose that Q lies in L^2_k so or that $Q = Q^*$, $Q \in L^r \cap L^{r'}$ for some r with $1 < r < 2$ and Q lies in L^2_k so that $I - G_zQ$ is invertible on $L^{\tilde{p}}_{-k}$. If $h \in L^{\tilde{p}}_{-k}(\mathbf{R}^2)$ and Q lies in $L^2_{k+1}(\mathbf{R}^2)$ with $C(p)\|Q\|_{L^2_k}$ small or $Q = Q^*$ so that $I - G_zQ$ is invertible on $L^{\tilde{p}}_{-k}$. Define m by*

$$m(x, z) = (I - G_zQ)^{-1}(h).$$

The function $z \to m(\cdot, z)$ is differentiable as a map from the complex plane into $L^{\tilde{p}}_{-k}$.

If $k\tilde{p} > 2$ and $h = I_2$, then we have that

$$\frac{\partial}{\partial \bar{z}} m(x, z) = m(x, \bar{z}) E_z^{-1} S(z).$$

Proof. From the Hölder inequality, we obtain that Qf is in $L^{\tilde{p}}_1$ and our hypotheses imply that $L^{\tilde{p}}_1 \subset L^1$. From Corollary 20.9, it follows that $I - G_zQ$ is strongly differentiable as an operator on $L^{\tilde{p}}_{-k}(\mathbf{R}^2)$. From Lemma A.5 of Appendix A, it follows that $m(\cdot, z) = (I - G_zQ)^{-1}(h)$ is differentiable as a map from $\mathbf{R}^2$ into $L^{\tilde{p}}_{-k}$.

If we let $h = I_2$, then $DI_2 = 0$ and it is clear that $m(\cdot, z) = (I - G_zQ)^{-1}(I_2)$ will satisfy the system (18.3). From Lemma A.5 the derivative with respect to $\bar{z}$ will be

$$\frac{\partial}{\partial \bar{z}} m(\cdot, z) = (I - G_zQ)^{-1}(\frac{\partial}{\partial \bar{z}} G_zQ)(I - G_zQ)^{-1}(I_2). \tag{20.11}$$

Since $m(\cdot, z) = (I - G_zQ)^{-1}(I_2)$, we have that

$$\frac{\partial}{\partial \bar{z}} G_z(Q(I - G_zQ)^{-1}(I_2)) = -\frac{2}{\pi} J \int A(x - y, z) Q(y) m(y, z)^d \, dy = E_z^{-1} S(z).$$

The quantity $S(z)$ is defined by

$$S(z) = -\frac{2}{\pi} J \int_{\mathbf{R}^2} E_z(Qm(\cdot, z)^d)^o dy \tag{20.12}$$

and $J = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$. We will see that S plays an important role in our theory and call S the *scattering data* associated to the potential Q. Note that from the weighted Hölder inequality, we have Qm is in $L^p_1 \subset L^1$. Hence, it follows that S is a bounded function. Thus we may rewrite (20.11) more compactly as

$$\frac{\partial}{\partial \bar{z}} m(\cdot, z) = (I - G_zQ)^{-1}(E_z^{-1} S(z)). \tag{20.13}$$

We now simplify the right-hand side. We observe that if S is an off-diagonal matrix that is independent of x, a calculation shows that $D_z(fE_z^{-1}S(z)) = (D_{\bar z}f)E_z^{-1}S(z)$. This implies that $G_z(fE_z^{-1}S(z)) = G_{\bar z}(f)E_z^{-1}S(z)$ whenever f in $L^p(\mathbf{R}^2)$ for some p between 1 and 2. As we have that m satisfies the equation $m(\cdot,\bar z) - G_{\bar z}(Qm(\cdot,\bar z)) = I_2$, we may right-multiply this equation by $E_z^{-1}S(z)$ and obtain that the quantity $m(\cdot,\bar z)E_z^{-1}S(z)$ satisfies the equation

$$m(\cdot,\bar z)E_z^{-1}S(z) - G_z(Qm(\cdot,\bar z)E_z^{-1}S(z)) = E_z^{-1}S(z).$$

Applying $(I - G_zQ)^{-1}$ to both sides yields

$$m(\cdot,\bar z)E_z^{-1}S(z) = (I - G_zQ)^{-1}(E_z^{-1}S(z)).$$

Hence we may rewrite (20.13)

$$\frac{\partial}{\partial \bar z} m(\cdot, z) = m(\cdot,\bar z)E_z^{-1}S(z). \tag{20.14}$$

This completes the proof of the Corollary.

∎

Exercise 20.15 *Show that, under the hypotheses of the above theorem $S(z)$ is continuous and* $\lim_{|z|\to\infty} S(z) = 0$.

20.3 Higher derivatives with respect to z

Next, we sketch the proof that our Jost solutions have additional derivatives with respect to z. Towards this end, we consider the iterated integral equation for m^d,

$$m^d(x,z) - GQG_zQm^d(x,z) = I_2.$$

We will prove by induction that this function is differentiable of all orders. We assume that m^d has ℓ derivatives with respect to z and these derivatives exist in the space $L^{\tilde p}_{-k}$. Let α be a multi-index of length ℓ. Differentiating the iterated integral equation with respect to z gives

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial z^\alpha} m^{11}(x,z) - \frac{1}{\pi^2}\int_{\mathbf{R}^4} \frac{q^{12}(x_1)a^2(x_2-x_1,z)q^{21}(x_2)}{(x-x_1)(\bar x_1 - \bar x_2)} \frac{\partial^\alpha}{\partial z^\alpha} m^{11}(x_2,z)\,dx_1\,dx_2 \\
&= \sum_{\beta+\gamma=\alpha,\gamma\neq\alpha} \frac{\alpha!}{\beta!\gamma!}\frac{1}{\pi^2}\int_{\mathbf{R}^4} \frac{q^{12}(x_1)(i(x_1-x_2))^\beta a^2(x_2-x_1,z)q^{21}(x_2)}{(x-x_1)(\bar x_1-\bar x_2)} \frac{\partial^\gamma}{\partial z^\gamma} m^{11}(x_2,z)\,dx_1\,dx_2 \\
&\equiv F(x,z).
\end{aligned} \tag{20.16}$$

We have defined the right-hand side of the previous equation to be $F(x,z)$ which satisfies

$$\|\frac{\partial}{\partial z^j}F(\cdot,z)\|_{L^{\tilde p}_{-k}} \leq C\|Q\|_{L^2_{\ell+1}}\|Q\|_{L^2_{k+\ell+1}}\sum_{|\gamma|\leq \ell}\|\frac{\partial^\gamma}{\partial z^\gamma}m^{11}(\cdot,z)\|_{L^{\tilde p}_{-k}}, \qquad j=1,2.$$

Corollary 20.9 implies that the operator on GQG_zQ is differentiable with respect to z as an operator in the strong topology for operators on $L^{\tilde p}_{-k}$. Hence, we may use Lemma A.5 from Appendix A to conclude that the solution of this integral equation has one more derivative with respect to z. This will require that Q lie in $L^2_{k+\ell+1}$ if m lies in $L^{\tilde p}_{-k}$.

We have more or less proven.

Theorem 20.17 *Suppose that $k \geq 0$ and $1 < p < 2$ with $k\tilde p > 2$. Suppose that Q lies in the Schwartz class and that either $Q = Q^*$ or that Q is small in L^2_k so that $I - G_zQG_zQ$ is invertible. Suppose that $m = (I - G_zQ)^{-1}(I_2)$ is the solution of (18.3) which lies in $L^{\tilde p}_{-k}$.*

Under these hypotheses, all derivatives of the diagonal part of m, m^d, will lie in $L^\infty(L^{\tilde p}_{-k})$.

Finally, combining the arguments used to prove Theorem 20.3 and Theorem 20.17 we can prove the following.

Theorem 20.18 *If Q lies in the Schwartz class and either $Q = Q^*$ or $\|Q\|_{L^2_k}$ is small for some $k > 0$, then the solution m^d is infinitely differentiable and for p and k so that $k > 2/\tilde p$ we have*

$$\frac{\partial^{\alpha+\beta}}{\partial x^\beta z^\alpha}m^d \in L^\infty(dz; L^{\tilde p}_{-k}).$$

Exercise 20.19 *Note that we only are asserting a result for the diagonal part of m, where the map E_z has no effect. Formulate and prove a similar result for the off-diagonal part of m.*

Proof. We begin by following the arguments in Theorem 20.17 to obtain that

$$\frac{\partial^\alpha}{\partial z^\alpha}m^d(\cdot,z) - GQG_zQ\frac{\partial^\alpha}{\partial z^\alpha}m^d(\cdot,z) = F(\cdot,z)$$

where F is as in (20.16). Examining (20.16), we see that $\sup_z\|\nabla_x F(\cdot,z)\|_{L^p} \leq C(Q)\sum_{|\gamma|\leq|\alpha|}\|\frac{\partial^\gamma}{\partial z^\gamma}m^d\|_{L^\infty_z(L^p)}$. Now, from Lemma 20.2, it follows that $\frac{\partial^\alpha}{\partial z^\alpha}m^d$ is differentiable once with respect to x and the derivative lies in $L^\infty(L^p)$.

Finally, an induction argument as in Theorem 20.3 gives higher derivatives with respect to x.

Note that the method from Theorem 20.3 gives that each derivative lies in L^p, but if the ℓth derivatives lie in $L^\infty(L^p)$, then Sobolev embedding gives that the derivatives of order $\ell - 1$ lie in $L^\infty(L^{\tilde{p}})$. ∎

We give the following result on the map which takes a potential Q to the scattering data S. Given a potential Q, we define the *scattering map* $\mathcal{T}$ by $\mathcal{T}(Q) = S$.

Theorem 20.20 *Let Q be an off-diagonal matrix-valued Schwartz function and suppose that either $\|Q\|_{L^2_k}$ is small for some $k > 0$ or that $Q = Q^*$. Then $\mathcal{T}(Q)$ lies in $\mathcal{S}(\mathbf{R}^2)$.*

Proof. We recall the definition of the scattering data (20.12)

$$S(z) = -\frac{2}{\pi} J \int_{\mathbf{R}^2} E_z(Qm^d)(y,z)\, dy.$$

Let k be a non-negative integer and suppose that β is a multi-index. We may write $z^k \frac{\partial^\beta}{\partial z^\beta} S(z)$ as

$$(-J)^{k+1} \frac{2}{\pi} \int_{\mathbf{R}^2} \Big(\sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} \frac{\partial^\gamma}{\partial z^\gamma} A(y,-z) D^k (Q(y) \frac{\partial^\delta}{\partial z^\delta} m^d(y,z)\Big)\, dy.$$

When Q is in the Schwartz class, the estimates for derivatives of m in Theorem 20.18 allow us to show that this integral is bounded. ∎

Exercise 20.21 *(Long!) Show that the map $\mathcal{T}$ is continuous on the Schwartz class.*

Chapter 21

Asymptotic expansion of the Jost solutions

In this section, we establish an asymptotic expansion for the Jost solutions, m. We continue to assume that the potential Q is a matrix-valued Schwartz function. We will base our expansion on the construction of m as a solution of (18.3). We observe that similar arguments, with the roles of x and z reversed, will give asymptotic expansions of the function m which solves the $\bar{\partial}$-equation, (20.14), below.

21.1 Expansion with respect to x

We begin by giving an expansion in the variable x. We recall our convention that $n = E_z m$. The first expansion is more natural in this variable.

We introduce mixed L^p spaces on $\mathbf{R}^4$. The space $L^p_z(L^q_x)$ will denote the collection of measurable functions on $\mathbf{R}^4$ for which the norm,

$$\|m\|_{L^p_z(L^q_x)} = \left(\int_{\mathbf{R}^2} \left(\int_{\mathbf{R}^2} |m(x,z)|^p \, dx \right)^{q/p} dz \right)^{1/q}$$

is finite. We make the usual extension to the case $p = \infty$.

Exercise 21.1 *Show that for $1 \leq p, q \leq \infty$, these spaces are Banach spaces.*

Theorem 21.2 *Suppose that Q is in the Schwartz class. Fix p with $1 < p < 2$ and k so that $k > 2/\tilde{p}$. Suppose that Q is small in L^2_k or that $Q = Q^*$. We have*

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{\ell} \left[n(x,z) - \sum_{j=0}^{\ell} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{-j} n_j(z) \right] \in L^\infty_z(L^{\tilde{p}}).$$

Here, $n_0 = I_2$ and for $j \geq 1$, we define

$$n_j(x,z) = \frac{1}{\pi} \int_{\mathbf{R}^2} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{j-1} (E_z Q m)(x,z)\, dy.$$

Proof. We proceed by induction. The base case follows from our theorem on existence which gives that $n - I_2 = G E_z Q m$. Now we consider

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix} (n(x,z) - I_2) = \frac{1}{\pi} \int \begin{pmatrix} \frac{x}{x-y} & 0 \\ 0 & \frac{\bar{x}}{\bar{x}-\bar{y}} \end{pmatrix} E_z Q m(y,z)\, dy$$

On the right-hand side of this expression, we may add and subtract y or $\bar{y}$ in the numerator to obtain

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix} (n(x,z) - I_2) = n_1(x,z) + \frac{1}{\pi} \int_{\mathbf{C}} \begin{pmatrix} \frac{y}{x-y} & 0 \\ 0 & \frac{\bar{y}}{\bar{x}-\bar{y}} \end{pmatrix} E_z Q m(y,z)\, dy.$$

Rearranging this last expression gives the next term of the asymptotic expansion.

It is now clear how to proceed by induction. We suppose that we have the expansion up to order ℓ with the exact error term

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{\ell} (n(x,z) - \sum_{j=0}^{\ell} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{-j} n_j(z)) = \frac{1}{\pi} \int_{\mathbf{C}} \begin{pmatrix} \frac{y^\ell}{x-y} & 0 \\ 0 & \frac{\bar{y}^\ell}{\bar{x}-\bar{y}} \end{pmatrix} E_z Q m(y,z)\, dy. \tag{21.3}$$

We multiply both sides by the matrix $\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}$ and then add and subtract y or $\bar{y}$ in the numerator of each term in the matrix to conclude

$$\begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{\ell+1} (n(x,z) - \sum_{j=0}^{\ell} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}^{-j} n_j(z)) = n_{\ell+1}(z) + \frac{1}{\pi} \int_{\mathbf{C}} \begin{pmatrix} \frac{y^{\ell+1}}{x-y} & 0 \\ 0 & \frac{\bar{y}^{\ell+1}}{\bar{x}-\bar{y}} \end{pmatrix} E_z Q m(y,z)\, dy.$$

Rearranging gives the expansion (21.3) with ℓ replaced by $\ell + 1$. The estimate (19.17) for m in $L^{\tilde{p}}_{-k}$, our hypothesis that Q is in the Schwartz class implies that $y^{\ell+1}Q(y)$ is in L^2_k and thus Corollary 18.8 implies that the right-hand side of the previous equation is in $L^\infty_z(L^{\tilde{p}}_x)$. ∎

Remark. Note that, up to a constant multiple, the term $n_1(z)$ is our scattering data that appears below in (20.12).

21.2 Expansion in z

In this section, we give a second expansion for the Jost solution m. The terms in this expansion will reappear when we study the Davey-Stewartson equation by an inverse scattering method.

We begin with a Lemma which is a slight extension of our results on invertibility of $(I - G_Z Q)$.

Lemma 21.4 *Let $k \geq 0$ and $1 < p < 2$ and suppose that Q lies in the Schwartz class. If either Q is small in L^2_k or $Q = Q^*$, then the map $I - G_z Q G_z Q$ is invertible on $L^{\tilde{p}}_{-k}$.*

Proof. In the case that Q is small, we may find the inverse with a Neumann series as in Corollary 18.11.

In the case where we assume $Q = Q^*$, we may use Lemma 19.11 to see the uniqueness. With Theorem 19.7, the invertibility on $L^{\tilde{p}}_{-k}$ follows from the Fredholm theory. To see that the map inverse is bounded on $L^\infty(L^{\tilde{p}}_{-k})$, we may use the smoothness of Q to obtain the operator norm on $L^{\tilde{p}}_{-k}$ is bounded for large z as in Theorem 19.16 and the continuity of the operator $(I - G_z Q G_z Q)^{-1}$ (see Lemma 18.10) to conclude the boundedness for all z. ∎

We give the expansion in z.

Theorem 21.5 *Supppose that Q is an off-diagonal matrix valued Schwartz function and that either $Q = Q^*$ or that Q is small in L^2_k for appropriate k. For $\ell = 0, 1, 2, \ldots,$ we have the following expansion for the Jost solution m,*

$$z^\ell (m(x,z) - \sum_{j=0}^{\ell} z^{-j} m_j(x)) \in L^\infty_z(L^{\tilde{p}}).$$

The coefficients m_j are given by $m_0 = I_2$, $m^o_1 = 2JQ$, $m^o_{j+1} = 2J(-D + QGQ)m^o_j$ and $m^d_j = GQ(m^o_j)$ for $j = 1, 2, \ldots$.

Proof. We begin our proof by observing that if we write

$$zA(y,z) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} DA(y,z) = -2JDA(y,z), \tag{21.6}$$

then we may integrate by parts in the expression $G_z(f^o)$ to obtain

$$zG_z(f^o) = 2Jf^o - G_z(2JDf^o). \tag{21.7}$$

Now we begin with the iterated integral equation for the off-diagonal part of the Jost solution, m. We have

$$(I - G_zQGQ)(m^o) = G_zQ.$$

We multiply by z and use (21.7) on the right to obtain

$$(I - G_zQGQ)(zm^o) = 2JQ - G_z(2JDQ).$$

The first term on the left is m_1^o. We subtract this to the other side and subtract G_zQGQm_1 from both sides which leads to

$$(I - G_zQGQ)(z(m^o - z^{-1}m_1^o) = G_z(-Dm_1^o + QGQm_1^o).$$

As the argument of G_z on the right is a Schwartz function, we may use the invertibility of the operator $I - G_zQGQ$ on $L^\infty(L^{\tilde{p}})$ to obtain the expansion for m^o and $\ell = 1$.

We now proceed by induction. Our induction hypothesis is that

$$(I - G_zQGQ)(z^\ell(m^o - \sum_{j=1}^{\ell} m_j^o z^{-j})) = G_z((-D + QGQ)m_\ell^o).$$

We repeat the manipulations from the first step. That is we begin by multiplying both sides by z and then use (21.7). This gives

$$(I - G_zQGQ)(z^{\ell+1}(m^o - \sum_{j=1}^{\ell} m_j^o z^{-j})) = m_{\ell+1}^o - G_z(Dm_{\ell+1}^o)$$

We subtract the expression $G_zQGQm_{\ell+1}^o$ from both sides and rearrange to obtain

$$(I - G_zQGQ)(z^{\ell+1}(m^o - \sum_{j=1}^{\ell+1} m_j^o z^{-j})) = G_z((-D + QGQ)m_{\ell+1}^o).$$

This completes the proof of the induction step. Applying the inverse of the operator $I - G_zQGQ$, we obtain that $z^{\ell+1}(m^o - \sum_{j=1}^{\ell+1} m_j^o z^{-j})$ lies in $L^\infty(L^{\tilde{p}})$.

To obtain the asymptotic expansion for the diagonal part, we write $m^d = I_2 + G_zQm^o$ and substitute the expansion for m^o. ∎

21.3 The inverse of the scattering map.

In this section, we sketch the proof that we have an inverse to the scattering map $\mathcal{T}$. To see this, we begin with the $\bar\partial$-equation (20.14). The form of this equation is similar to our equation for m in the x variable (18.3) and we shall see that we may adapt the techniques used to study (18.3) to obtain a second representatation of the Jost solution m as

$$m = (I - gT)^{-1}(I_2) \tag{21.8}$$

Here, we are using T to denote the map $Tm(x,z) = m(x,\bar z)S(z)A(x,-\bar z)$ which appears in the $\bar\partial$ equation, (20.14) and g denotes the Cauchy transform acting matrix-valued functions of the variable z.

Exercise 21.9 *Let $m = (I - gT)^{-1}$ defined in (21.8). Give a formal calculation which shows that m is a solution of (18.3) and find an expression for Q in terms of S.*

Hint: The answer is

$$Q(x) = \frac{1}{\pi}\mathcal{J}\int_{\mathbf{R}^2} Tm(x,z)\,dz.$$

Here, $\mathcal{J}$ acts on 2×2 matrices by $\mathcal{J}a = 2Ja^o = -2a^oJ = [J,a]$ where $[J,a]$ is the commutator and the matrix J is as in the proof of Corollary 20.10.

Exercise 21.10 *Justify each step in the previous exercise.*

We want to establish a global existence result for the equation (20.14). To do this, we will need a substitute for the condition $Q = Q^*$ that we used in studying (18.3). This substitute is given in the next Lemma.

Lemma 21.11 *Suppose Q satisfies one of our standard conditions for uniqueness. With $\mathcal{T}$ the scattering map defined by (20.12), we have $\mathcal{T}(\pm Q^*)(z) = \pm S^*(\bar z)$.*

Proof. We define two involutions on matrix valued functions on $\mathbf{R}^4$ by

$$\mathcal{U}^{\pm}f(x,z) = \pm\begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}\bar m(x,\bar z)\begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}.$$

Straightforward calculations show that

$$\mathcal{U}^{\pm}(D_z m) = D_z\mathcal{U}^{\pm}m, \qquad \frac{\partial}{\partial\bar z}\mathcal{U}^{\pm}m = \mathcal{U}^{\pm}\frac{\partial}{\partial\bar z}m, \qquad \text{and} \qquad \mathcal{U}^{\pm}(Qm) = \pm Q^*\mathcal{U}^{\pm}m.$$

Thus, if m is the Jost solution to (18.3) for the potential Q, we may apply $\mathcal{U}^{\pm}$ to both sides of the equation $D_z m - Qm$ and obtain

$$D_z \mathcal{U}^{\pm} m = \pm Q^* \mathcal{U}^{\pm} m.$$

Also, $\mathcal{U}^{\pm} m - I_2 \in L^p(\mathbf{R}^2)$ for every z. Thus, by uniqueness, $\mathcal{U}^{\pm} m$ is the Jost solution for $\pm Q^*$. Using this in the expression for the scattering data (20.12), we conclude that

$$\begin{aligned} \mathcal{T}(\pm Q^*)(z) &= \frac{-\mathcal{J}}{\pi} \int_{\mathbf{R}^2} E_z(\pm Q^* \mathcal{U}^{\pm} m(x,z))\, dx \\ &= \frac{-2J}{\pi} \int_{\mathbf{R}^2} \begin{pmatrix} 0 & \bar{q}^{21}(x)\bar{m}^{11}(x,\bar{z})a^1(x,-z) \\ \bar{q}^{12}(x)\bar{m}^{22}(x,\bar{z})a^2(x,-z) & 0 \end{pmatrix} dx. \end{aligned}$$

We have that $a^1(x,-z) = \bar{a}^2(x,-\bar{z})$ and $a^2(x,-z) = \bar{a}^1(x,-\bar{z})$ and substituting this into the previous equation gives that

$$\mathcal{T}(\pm Q^*)(z) = \pm \mathcal{T}(Q)^*(\bar{z}).$$

∎

Remark. Using each of these formulae in succession gives that $\mathcal{T}(-Q) = -\mathcal{T}(Q)$.

Lemma 21.12 *Suppose that S is an off-diagonal matrix-valued function with entries in $\mathcal{S}(\mathbf{R}^2)$ and satisfying $S(z) = S^*(\bar{z})$. Suppose m is a function in $L^{\tilde{p}}$ for some $p < \infty$, and m is a solution of the equation*

$$\frac{\partial}{\partial \bar{z}} m(z) = m(\bar{z}) A(x,z) S(z)$$

for some x. Then $m = 0$.

Proof. Our proof is essentially the same as Theorem 19.6. We let $u^{\pm}(z) = m^{11}(z) \pm \bar{m}^{12}(\bar{z})$ and $v^{\pm}(z) = m^{21}(z) \pm \bar{m}^{22}(\bar{z})$. Using that

$$\frac{\partial}{\partial \bar{z}} \bar{w}(\bar{z}) = \overline{\frac{\partial w}{\partial \bar{z}}(\bar{z})},$$

a straightforward calculation shows that

$$\frac{\partial}{\partial \bar{z}} u^{\pm}(z) = a^2(x,z) S^{21}(z) \bar{u}^{\pm}(z)$$

and for $v^{\pm}$, we have

$$\frac{\partial}{\partial \bar{z}} v^{\pm}(z) = a^2(x,z) S^{21}(z) \bar{v}^{\pm}(z).$$

In both cases, our uniqueness theorem for pseudo-analytic functions, Theorem 19.4, imply that $u^{\pm} = v^{\pm} = 0$. ∎

With this lemma, we can prove the injectivity of the operator $I - gT$ and use the Fredholm theory to find $m = (I - gT)^{-1}(I_2)$ when we have $S(z) = S^*(\bar{z})$ and say S is in the Schwartz class. Of course, existence also follows when we have S small in L^2_k.

Theorem 21.13 *Suppose that Q is in the Schwartz class and $Q = Q^*$. Then the map $\mathcal{T}$ is injective on the set $\{Q : Q = Q^*,\ Q^d = 0,\ Q \in \mathcal{S}(\mathbf{R}^2)\}$.*

Proof. According to Theorem 20.20, we have that if Q is an self-adjoint off-diagonal matrix valued function in $\mathcal{S}$, then $S = \mathcal{T}(\mathcal{Q})$ is an off-diagonal matrix-valued function satisfying $S^*(\bar{z}) = S(z)$. Suppose we have two potentials Q_1 and Q_2 for which $\mathcal{T}(Q_1) = \mathcal{T}(Q_2) = S$. According to Theorem 20.20 we have that S is a matrix-valued Schwartz function. Let m_1 and m_2 denote the two Jost solutions. By the Corollary at the end of Chapter 18 (not in the notes, yet), we have that $m_1 - m_2$ is in $L^{\tilde{p}}(\mathbf{R}^4)$, thus for almost every x, we may use the $\bar{\partial}$-equation (20.14) and Lemma 21.12 to conclude that $m_1 - m_2$ is zero. From Theorem 21.5, if $m_1 = m_2$, we have that $Q_1 = Q_2$. ∎

Exercise 21.14 *Show that $\mathcal{T}$ is injective for potentials which are in the Schwartz class and which are small in L^2_k.*

Our last result is to show that the map $\mathcal{T}$ is onto.

Theorem 21.15 *The map $\mathcal{T}$ maps $\{Q : Q = Q^*,\ Q \in \mathcal{S}(\mathbf{R}^2)\}$ onto the set of matrix-valued, off-diagonal Schwartz functions which satisfy $S^*(z) = S(\bar{z})$.*

Proof. Let S be a matrix valued Schwartz function satisfying $S^*(z) = S(\bar{z})$. We may imitate our study of solutions to the equation (18.3) to construct m which satisfies (20.14). As in exercise 21.10, we can show that the Jost solution m satisfies (18.3) for some potential Q. Imitating the (very long) proof of Theorem 20.20 gives that the potential Q is in $\mathcal{S}$. ∎

Exercise 21.16 *Show that $\mathcal{T}$ is onto the intersection of neighborhood of zero in L^2_k with the Schwartz class.*

Chapter 22

The scattering map and evolution equations

In this chapter, we develop further properties of the scattering map defined above and show how this scattering map can be used to study a non-linear evolution equation. The non-linear equation presented here was known prior to its treatment by the method of inverse scattering. The equation first arose in the study of water waves.

Our first result gives a remarkable identity for the scattering transform which may be viewed as a type of non-linear Plancherel identity for the scattering map $\mathcal{T}$. This identity was fundamental in the treatment of the inverse conductivity problem in two dimensions by the author and Uhlmann [9]. However, recent work of Astala and Päivärinta have provided a better treatment in two dimensions [3].

Next, we find the linearization of the scattering map $\mathcal{T}$ and its inverse. We use this to help find a family of evolution equations which may be treated by the inverse scattering method. The treatment we follow is taken from Beals and Coifman [4]. These results may also be found in Ablowitz and Fokas [12].

22.1 A quadratic identity

We begin with a remarkable identity for the map $\mathcal{T}$. To motivate this result, we observe that as the potential tends to zero in L^2_k, examining the series for the Jost solution in Chapter 18 gives that the Jost solution m converges to I_2.

Thus, if we consider the linearization at 0, we find that for sufficiently nice potentials

Q,

$$\lim_{\epsilon\to 0^+} \frac{\mathcal{T}(\epsilon Q) - \mathcal{T}(0)}{\epsilon} = \frac{-2}{\pi} J \begin{pmatrix} 0 & \hat{q}^{12}(2\bar{z}) \\ \hat{q}^{21}(-2z) & 0 \end{pmatrix}.$$

Thus, this linearized map will satisfy some variant of the Plancherel identity. We will show below that we have a similar identity for the non-linear map.

To give the proof of our identity, we start with the definition of the scattering map,

$$S(z) = \frac{-1}{\pi}\mathcal{J}\int E_z(Qm)\,dx$$

and the definition of the inverse map

$$Q(x) = \frac{1}{\pi}\mathcal{J}\int m(x,\bar{z})S(z)A(x,-\bar{z})\,dx$$

which may be found in (20.12) and Exercise 21.10.

Thus, using (20.12) and Exercise 21.10, we obtain

$$\begin{aligned}\int_{\mathbf{R}^2} \operatorname{tr} Q(x)^2\,dx &= \frac{1}{\pi}\operatorname{tr}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} 2Jm^d(x,\bar{z})S(z)A(x-\bar{z})Q(x)\,dz\,dx \\ &= \frac{1}{\pi}\operatorname{tr}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} S(z)(-2J)A(x,-\bar{z})Q(x)m^d(x,\bar{z})\,dx\,dz \\ &= \int_{\mathbf{R}^2}\operatorname{tr} S(z)S(\bar{z})\,dz. \qquad (22.1)\end{aligned}$$

In the second equality, we apply the simple identity

$$\begin{pmatrix} \alpha^{11} & 0 \\ 0 & \alpha^{22} \end{pmatrix}\begin{pmatrix} 0 & \beta^{12} \\ \beta^{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta^{12} \\ \beta^{21} & 0 \end{pmatrix}\begin{pmatrix} \alpha^{22} & 0 \\ 0 & \alpha^{11} \end{pmatrix}.$$

First, to commute $2Jm^d$ and S and then to commute m^d and Q. Theorem 20.20 which implies that S is in the Schwartz class and our estimates for the Jost solutions m in Theorem 19.8 justify the use of Fubini's theorem.

With this Lemma, we have the following Theorem.

Theorem 22.2 *Suppose Let $Q \in \mathcal{S}$ and suppose that either $\|Q\|_{L^2_k}$ is small and $Q = -Q^*$ or that $Q = Q^*$. With $S = \mathcal{T}(Q)$, we have that*

$$\int |Q(x)|^2\,dx = \int |S(z)|^2\,dz.$$

Proof. If $Q = Q^*$, then $\mathcal{T}(Q) = \mathcal{T}(Q^*)$ and hence, from Lemma 21.11, $\mathcal{T}(Q)(z)^* = \mathcal{T}(Q)(\bar{z})$. The conclusion of our theorem follows from (22.1).

When $Q = -Q^*$, then Lemma 21.11 implies that $\mathcal{T}(Q)(z) = -\mathcal{T}(Q)(\bar{z})^*$ and thus $S(\bar{z}) = -S(z)^*$. Now, using the identity (22.1) and our Lemma 21.11, we obtain that

$$\int |Q(x)|^2\,dx = -\int \operatorname{tr}(Q(x)Q(x))\,dx = -\int \operatorname{tr} S(z)S(\bar{z})\,dz = \int |S(z)|^2\,dz.$$

∎

Exercise 22.3 *(Open.) Show that the map $\mathcal{T}$ is continuous in the L^2-norm.*

22.2 The tangent maps

In this section, we look at the linearization of the map $\mathcal{T}$ or the tangent map. Thus, if $Q(t)$ is a curve in the Schwartz space, we will try to differentiate $S(t) = \mathcal{T}(Q(t))$ with respect to t. We also will want to carry out the same exercise for $Q(t) = \mathcal{T}(S(t))$. To begin, we introduce some more notation. Given a potential Q, we let $\tilde{Q} = -Q^t$ where Q^t is the ordinary transpose of a matrix and then $\tilde{m}$ will be the Jost solution for the potential $\tilde{Q}$. We will use two matrix-valued forms

$$\langle f, g\rangle_1 = \frac{1}{\pi}\int (f(x,z)^t g(x,z))^o\,dx \qquad \text{and} \qquad \langle f, g\rangle_2 = \frac{1}{\pi}\int (f(x,z)g(x,-z)^t)^o\,dz.$$

Note that $\langle f, g\rangle_1$ will be a matrix valued function of x and $\langle \cdot, \cdot\rangle_2$ will be a matrix-valued function of z. We observe that the formal transpose of Q with respect to form $\langle \cdot, \cdot\rangle_1$ is $-\tilde{Q}$. Thus, $\langle Qf, g\rangle_1 = -\langle f, \tilde{Q}g\rangle_1$. If we let D_z^τ denote the transpose of D_z with respect to the form $\langle \cdot, \cdot\rangle_1$, we have

$$D_z^\tau(fA_{-z}) = -(D_{-\bar{z}}f)A_{-z}. \tag{22.4}$$

Next, we observe that under our standard hypotheses, we have that $(I - QG_z)^{-1}$ is an invertible map on the dual of $L^{\tilde{p}}_{-k}$, $L^{\tilde{p}'}_k$.

Lemma 22.5 *Let Q satisfy our standard hypotheses. Then the operator $(I - QG_z)$ is invertible on $L^{\tilde{p}'}_k$ and we have the identity*

$$Q(I - G_zQ)^{-1} = (I - QG_z)^{-1}Q.$$

Proof. Under our standard hypotheses, we have that $(I - QG_z)^{-1}$ is an invertible map on dual of $L^{\tilde{p}}_{-k}$, $L^{\tilde{p}'}_{k}$. To see this, observe that with respect to the bi-linear form $\int \operatorname{tr}(f^t g)\,dx$, the transpose of Q is Q^t and the transpose of G_z is $-G_{-z}$. Thus we have that $(I - QG_z)^{-1}$ is the transpose of $(I - G_{-z}\tilde{Q})^{-1}$ and our assumptions on Q guarantee that the operator $I - G_{-z}\tilde{Q}$ is invertible.

To establish the second part, we can begin with the identity

$$(I - QG_z)Q = Q(I - G_zQ)$$

and then apply $(I - G_zQ)^{-1}$ on the right and $(I - QG_z)^{-1}$ on the left. ∎

We now find the relation between $\mathcal{T}(Q)$ and $\mathcal{T}(\tilde{Q})$. This is of some interest and the techniques we develop are needed to find the tangent maps.

We begin with the definition of the scattering map, which may be expressed using the form $\langle \cdot, \cdot \rangle_1$ as

$$\begin{aligned}
\mathcal{T}(Q)(z) &= -\mathcal{J}\langle A_{-z}, Qm\rangle_1 \\
&= -\mathcal{J}\langle A_{-z}, (I - QG_z)^{-1}(QI_2)\rangle_1 \\
&= \mathcal{J}\langle \tilde{Q}(I + G_z^\tau \tilde{Q})^{-1}(A_{-z}), I_2\rangle_1 \\
&= \mathcal{J}\langle (\tilde{Q}(I - G_{-\bar{z}}\tilde{Q})^{-1}(I_2))A_{-z}, I_2\rangle_1 \\
&= \mathcal{J}\langle (\tilde{Q}\tilde{m}(\cdot, -\bar{z})A_{-z}, I_2\rangle_1 \qquad (22.6)
\end{aligned}$$

In this calculation, we have used F^τ for the transpose of an operator F with respect the form $\langle \cdot, \cdot \rangle_1$. The second equality uses Lemma 22.5. The fourth equality depends on (22.4) to rewrite the transpose of G_z^τ.

Now, we compute the action of the map $\mathcal{T}$ on the potential $\tilde{Q}$. We have

$$\begin{aligned}
\mathcal{T}(\tilde{Q}) &= -\mathcal{J}\langle A_{-z}, \tilde{Q}\tilde{m}\rangle_1 \\
&= -\mathcal{J}(\langle (\tilde{Q}\tilde{m}), A_{-z}\rangle_1)^t \\
&= +(\mathcal{J}\langle \tilde{Q}\tilde{m}, A_{-z}\rangle_1)^t \\
&= +(\mathcal{J}\langle \tilde{Q}\tilde{m}A_{\bar{z}}, I_2\rangle_1)^t \\
&= S(-\bar{z})^t.
\end{aligned}$$

Where the last equality depends on (22.6). Thus, we have

$$\mathcal{T}(\tilde{Q})(z) = \mathcal{T}(Q)(-\bar{z})^t.$$

Our next goal is to compute the tangent maps for $\mathcal{T}$ and $\mathcal{T}^{-1}$. Thus, we suppose that we have a curve $t \to Q(t)$ where $Q(t)$ is in the Schwartz class and we have that the time derivative $\dot{Q}$ exists in L^2_k. Thus we have

$$\lim_{h\to 0^+} \|\frac{Q(t+h)-Q(t)}{h} - \dot{Q}\|_{L^2_k} = 0.$$

Now if we note that the estimate for the operator G_zQ and use Lemma A.5, we obtain that

$$(\dot{Qm}) = \dot{Q}m + Q(I-G_zQ)^{-1}G_z\dot{Q}m = (I-QG_z)^{-1}(\dot{Q}m).$$

The last equality uses the identity that $I + B(I-AB)^{-1}A = (I-BA)^{-1}$.

Thus, with $S = \mathcal{T}(Q)$, we have

$$\begin{aligned}
\dot{S}(z) &= -\mathcal{J}\langle A_{-z}, (\dot{Qm})\rangle_1 \\
&= -\mathcal{J}\langle A_{-z}, (I-QG_z)^{-1}(\dot{Q}m)\rangle_1 \\
&= -\mathcal{J}\langle (I-G_z^\tau Q^\tau)^{-1}(A_{-z}), (\dot{Q}m)\rangle_1 \\
&= -\mathcal{J}\langle (I-G_{-\bar{z}}\tilde{Q})^{-1}(I_2)A_{-z}, \dot{Q}m\rangle_1 \\
&= -\mathcal{J}\langle \tilde{m}A_{-z}, \dot{Q}m\rangle_1
\end{aligned}$$

Thus, we have proven that if $Q(t)$ is a curve in $\mathcal{S}$ which satisfies our standard hypotheses for the invertibility of $(I - G_zQ)$, and is differentiable in L^2_k, then we have that

$$\dot{\mathcal{T}}Q(z) = -\mathcal{J}\langle \tilde{m}A_{-z}, \dot{Q}m\rangle_1.$$

Next, we consider the tangent map to the map $\mathcal{T}^{-1}$. To do this, we will recall T which is defined using the scattering data S by $Tm(x,z) = m(x,\bar{z})S(z)A(x,-\bar{z})$. Now assuming that S is a curve in the Schwartz class which is differentiable in L^2_k, we may differentiate m to obtain

$$\dot{m} = (I-gT)^{-1}(g\dot{T})(I-gT)^{-1}(I_2) = g(I-Tg)^{-1}(\dot{T}m). \tag{22.7}$$

Here, we have used the operator identity in Lemma 22.5 Differentiating Tm and using (22.7) we obtain

$$(\dot{Tm}) = \dot{T}m + T\dot{m} = \dot{T}m + Tg(I-Tg)^{-1}(\dot{T}m) = (I-Tg)^{-1}(\dot{T}m). \tag{22.8}$$

Then according to Exercise 21.10, we have

$$\mathcal{T}^{-1}(S)(x) = \mathcal{J}\langle Tm, I_2\rangle_2$$

Differentiating the right-hand side with respect to t and using our expression for $(\dot{Tm})$ in (22.8), we obtain

$$\dot{S} = \mathcal{J}\langle (I - Tg)^{-1}(\dot{T}m), I_2\rangle_2.$$

From here, we use that transpose of T with respect to the form $\langle \cdot, \cdot \rangle_2$ is $\tilde{T}$ defined using $\tilde{S}(z) = S(-\bar{z})^t$ and the transpose of g is again g. Thus, we obtain the tangent map for $\mathcal{T}^{-1}$,

$$\dot{Q} = \langle \dot{T}m, \tilde{m}\rangle_2 \tag{22.9}$$

We have proven.

Theorem 22.10 *Let $t \to Q(t)$ be a matrix-valued function defined on an interval in the real line. Suppose that for each t, $Q(t)$ lies in $\mathcal{S}$ and either $\|Q\|_{L^2_k}$ is small for each t or that $Q = Q^*$ for each t. If Q is differentiable as a map from the real line into L^2_k, then we have that $S(t) = \mathcal{T}(Q(t))$ is (pointwise) differentiable with respect to t.*

Similarly, if $S(t)$ lies in $\mathcal{S}$ and either $\|S\|_{L^2_k}$ is small or $S(z) = S^*(\bar{z})$ and S is differentiable as a function in L^2_k, then $Q(t) = \mathcal{T}^{-1}(S)(t)$ is (pointwise) differentiable with respec to t.

Exercise 22.11 *Show that $\dot{S}$ lies in the Schwartz class.*

Exercise 22.12 *If $t \to Q(t)$ is differentiable as a map into the Schwartz class, is $\mathcal{T}Q$ differentiable as a map into the Schwartz class?*

22.3 The evolution equations

Finally, we show how the map $\mathcal{T}$ is connected to a family of evolution equations. Thus, suppose that ϕ is a diagonal matrix-valued function and let $\Phi(f)(x, z) = f(x, z)\phi(z)$. Let T be the operation in the $\bar{\partial}$ equation, $Tm(x, z) = m(x, z)A(x, z)S(z)$ for some off-diagonal function S.

We let $\dot{T}$ be the derivative as operator on $L^{\tilde{p}}_{-k}$ and consider the evolution equation

$$\dot{T} = [\Phi, T] \tag{22.13}$$

where $[A, B] = AB - BA$ is the commutator of the operators A and B.

If we write out $[\Phi, T]$, we obtain

$$[\Phi, T] = f(x, \bar{z})\phi(\bar{z})S(z)A(x, -\bar{z}) - f(x, \bar{z})S(z)A(x, -\bar{z})\phi(z).$$

As the diagonal matrices A and ϕ commute, we have that the operator equation (22.13) is equivalent to the evolution for the function S

$$\dot{S}(z) = \phi(\bar{z})S(z) - S(z)\phi(z).$$

If we have that the time derivative $\dot{S}$ exists in L^2_k, then the operator T will be differentiable as an operator on $L^{\tilde{p}}_{-k}$, for example.

If we write $Q = \mathcal{T}^{-1}(S) = \langle Tm, I_2\rangle_2$, then we have shown in (22.9) that

$$\begin{aligned} \dot{Q} &= \mathcal{J}\langle \dot{T}m, \tilde{m}\rangle_2 \\ &= \mathcal{J}\langle (Tm)\phi, \tilde{m}\rangle_2 - \mathcal{J}\langle T(m\phi), \tilde{m}\rangle_2 \\ &= \mathcal{J}\langle (Tm)\phi, \tilde{m}\rangle_2 - \mathcal{J}\langle (m\phi), \tilde{T}\tilde{m}\rangle_2 \end{aligned}$$

where the last equality uses that the transpose of T with respect to $\langle\cdot,\cdot\rangle_2$ is $\tilde{T}$ defined using the scattering data $\tilde{S}$ associated with $\tilde{m}$. If we assume that ϕ has polynomial growth at infinity, then integrals in the above are well-defined when the scattering data S and $\tilde{S}$ are in $\mathcal{S}(\mathbf{R}^2)$. We will write $Tm = \frac{\partial}{\partial\bar{z}}m$ and $\tilde{T}\tilde{m} = \frac{\partial}{\partial\bar{z}}\tilde{m}$ and then integrate by parts. As the leading term of m and $\tilde{m}$ is the matrix I_2 we have to be careful if we want our integrals to make sense. Thus we introduce a cutoff function η_R, which is one if $|z| < R$ and 0 for $|z| > 2R$. Thus, we have

$$\begin{aligned} \dot{Q} &= \lim_{R\to\infty} \mathcal{J}\langle (\frac{\partial}{\partial\bar{z}}m)\eta_R\phi, \tilde{m}\rangle_2 - \mathcal{J}\langle m\phi\eta_R, \frac{\partial}{\partial\bar{z}}\tilde{m}\rangle_2 \\ &= \lim_{R\to\infty} \mathcal{J}\langle m\frac{\partial}{\partial\bar{z}}(\eta_R\phi), \tilde{m}\rangle_2 \end{aligned} \tag{22.14}$$

where we have integrated by parts. In the special case that $\phi(z) = z^k I_2$, the last limit becomes

$$\lim_{R\to\infty} \mathcal{J}\langle m\frac{\partial}{\partial\bar{z}}(\eta_R\phi), \tilde{m}\rangle_2 = [\mathcal{J}m(x,z)\tilde{m}(x,-z)^t]_{k+1}.$$

Where $[\mathcal{J}m(x,z)\tilde{m}(x,-z)^t]_\ell$ denotes the coefficient of $z^{-\ell}$ in the asymptotic expansion of $\mathcal{J}m(x,z)\tilde{m}(x,-z)^t$.

If we let

$$S(z,0) = \begin{pmatrix} 0 & s(z) \\ \bar{s}(\bar{z}) & 0 \end{pmatrix}$$

then for $\phi(z) = z^k$, we have that $S(z,t) = \exp(it(\bar{z}^k - z^k))S(z,0)$.

When $k = 2$, a rather lengthy calculation using the asymptotic expansion of Theorem 21.5 gives that

$$Q(x,t) = \mathcal{T}(S)(t) = \begin{pmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{pmatrix}$$

where

$$\begin{cases} \dot{q} = i(\frac{\partial^2}{\partial x_1 \partial x_2} q - 4q\phi) \\ \Delta\phi = \frac{\partial^2}{\partial x_1 \partial x_2} |q|^2 \end{cases}$$

Thus, to solve this system with a specified initial value, $q(x)$, we may write

$$Q(x,t) = \mathcal{T}^{-1}(\exp(-4itz^1z^2)\mathcal{T}(\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}))$$

and then the solution $q(x,t)$ will be the 1,2 entry of Q.

We could go on to talk about Lax pairs, the infinite-dimensional Hamiltonian structure and lots more. But not this year.

Exercise 22.15 *Find the evolutions corresponding to z^k for $k = 0$ and 1.*

Appendix A

Some functional analysis

A.1 Topologies

Below we summarize a few results from functional analysis.

We begin by recalling the standard topologies for linear operators between Banach spaces. If B and C are Banach spaces, we let $\mathcal{L}(B, C)$ denote the collection of all continuous linear maps from B to C. If $T \in \mathcal{L}(B, C)$, then we define

$$\|T\|_{\mathcal{L}(B,C)} = \sup\{\|Tx\|_C : x \in B\|x\|_B \leq 1\}.$$

It is well-known that $\|\cdot\|_{\mathcal{L}(B,C)}$ is a norm. The topology induced by this norm is called the norm topology.

However, there are two weaker topologies which are commonly used. The first is called the strong topology. This is the coarsest topology which makes the maps $T \to Tx$ continuous for each x in B. There is also the weak topology which is the coarsest topology for which we have that $T \to \lambda(Tx)$ is continuous for all $\lambda \in C^*$ and $x \in B$.

We are interested in operator valued functions $z \to T_z$ where z is a complex parameter and $T_z \in \mathcal{L}(B, C)$ for each z. We say that an operator valued function T_z is differentiable in the strong operator topology if we have partial derivatives $\partial T_z/\partial z$ and $\partial T_z/\partial \bar{z}$ which lie in $\mathcal{L}(B, C)$ and so that for each f in C,

$$\lim_{h \to 0} \frac{1}{|h|} \|T_{z+h} f - T_z f - h \frac{\partial T_z}{\partial z} f - \bar{h} \frac{\partial T_z}{\partial \bar{z}} f\|_C = 0.$$

A.2 Compact operators

Our goal here is to state the Fredholm theorem for operators on a Banach space X. We begin by defining compact operators. An operator $T : X \to Y$ on a Banach space is said to be *compact* if whenever $\{x_i\}$ is a bounded set in X, then $\{Tx_i\}$ contains a convergent subsequence.

We state a few elementary properties of compact operators.

Lemma A.1 *If T is a compact and S is bounded, then TS and ST are compact.*
The set of compact operators is closed in the operator norm.

Exercise A.2 *Prove the above Lemma.*

Exercise A.3 *If the image of T is finite dimensional, then T is compact.*

We state a version of the Fredholm alternative.

Theorem A.4 *If $T : X \to X$ is a compact operator, then ...*

A.3 Derivatives

Our main goal is the following Lemma which gives conditions that guarantee that the family of inverse operators is differentiable.

Lemma A.5 *Let T_z be a family of operators in $\mathcal{L}(B)$ and suppose that $z \to T_z$ is a family of operators which is differentiable at z_0 in the strong operator topology, that T_z^{-1} exists for z near z_0 and that the operator norms $\|T_z^{-1}\|$ are uniformly bounded for z near z_0. Then, the family T_z^{-1} is differentiable at z_0 and we have*

$$\frac{\partial}{\partial z}T_{z_0}^{-1} = -T_{z_0}^{-1}(\frac{\partial}{\partial z}T_{z_0})T_{z_0}^{-1} \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}}T_{z_0}^{-1} = -T_{z_0}^{-1}(\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}.$$

Proof. The proof is more or less the same as the proof of the quotient rule in calculus. We begin by observing that under these conditions, we have that $z \to T_z^{-1}$ is continuous at z_0 as a map from the complex plane into operators in the strong operator topology. Towards this end, we write

$$(T_{z_0+h}^{-1} - T_{z_0}^{-1})f = T_{z_0+h}^{-1}(T_{z_0} - T_{z_0+h})T_{z_0}^{-1}f.$$

From the strong continuity of the family T_z, we have that $\lim_{h\to 0}\|(T_{z_0}-T_{z_0+h})T_{z_0}^{-1}f\|_B = 0$. Our hypothesis that the norm of T_z^{-1} is bounded near z_0 allows us to conclude that

$$\lim_{h\to 0}\|T_{z_0+h}^{-1}(T_{z_0}-T_{z_0+h})T_{z_0}^{-1}f\|_B = 0.$$

Thus, $z \to T_z^{-1}$ is strongly continuous at z_0.

To establish the differentiability, we begin by writing

$$\begin{aligned} & T_{z_0+h}^{-1}f - T_{z_0}^{-1}f + T_{z_0}^{-1}(h\frac{\partial}{\partial z}T_{z_0} + \bar{h}\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}f \\ = \quad & (T_{z_0}^{-1} - T_{z_0+h}^{-1})(h\frac{\partial}{\partial z}T_{z_0} + \bar{h}\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}f \\ & \quad + T_{z_0+h}^{-1}(T_{z_0+h} - T_{z_0} + h\frac{\partial}{\partial z}T_{z_0} + \bar{h}\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}f. \end{aligned}$$

Recalling the strong continuity of T_z^{-1}, it follows that

$$\lim_{h\to 0}\frac{1}{|h|}\|(T_{z_0+h}^{-1} - T_{z_0}^{-1})(h\frac{\partial}{\partial z}T_{z_0} + \bar{h}\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}f\|_B = 0.$$

The differentiability of T_z implies that

$$\lim_{h\to 0}\frac{1}{|h|}\|T_{z_0}^{-1}(T_{z_0+h} - T_{z_0} + h\frac{\partial}{\partial z}T_{z_0} + \bar{h}\frac{\partial}{\partial \bar{z}}T_{z_0})T_{z_0}^{-1}f\|_B = 0.$$

The differentiability of T_z^{-1} follows from the previous displayed equations. ∎

Index

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