Recitation 10 MA113:004–006
29 September 1998 Fall 1998

Reminders: Quiz and homework due on Wednesday.
Below is a selection of problems related to sections 2.2 and 2.3. These problems will not be collected or graded. However, you should understand how to work each of these problems. You should begin working on these problems in groups in recitation. You will probably want to finish these problems outside of class. If you have questions, please ask your TA or instructor. If you find a problem difficult, consider working similar problems from the text for additional practice.

1. §2.2 #1,2,3,5,7,9,11,15
2. §2.3, #1,2,3,5,9,11,13,15,17,19,37,38.
3. Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 0} h(x)$ exists. Does $\lim_{x\to 0} g(x)$ always exist? Can you give an example where the limit of g does not exist? Why does the squeeze theorem not apply in this problem?
4. Use the principle of mathematical induction to prove that for $n = 1, 2, \ldots$,
$$\lim_{x\to a} x^n = a^n.$$
(Hint: The case $n = 1$ is the fundamental limit, #8 in the text. To go from n to $n + 1$, use the property 4 about multiplying limits.)
5. In this problem, we want to use the squeeze theorem to study the function $f(x)$ whose domain is $(0, \infty)$ and which satisfies
$$f(x)^2 = x \qquad \mathrm{f}(x) > 0.$$
(Sharp eyed students will note that this function is the positive square root function which is covered by property 10 on page 113.) However, we want to see how to prove 10 using just the two properties of f displayed above.)
   (a) Let $a > 0$ and consider the one-sided limit
   $$\lim_{x\to a^+} f(x) - f(a).$$
   If we multiply $f(x) - f(a)$ by $(f(x) + f(a))^{-1}$, simplify and conclude that, if $x > a$, then
   $$0 \leq f(x) - f(a) \leq \frac{x - a}{f(a)}.$$
   (Hint: Since $f(x) > 0$, we have $f(x) + f(a) > f(a)$. What happens if to the inequality if we take reciprocals and multiply by a positive number? If necessary, refer to the rules for inequalities in appendix A, page A2.)

(b) What happens if we apply the squeeze theorem to the one-sided limits in the previous part? (Sharp eyed students will note that the squeeze theorem in the text is not stated for one-sided limits. However, the theorem is still true if we replace the two-sided limits in the theorem by either of the one-sided limits.)

(c) Can you find a similar argument to show that the limit from below of $f(x)$ is also $f(a)$?

(d) Now that we know $\lim_{x\to a^+} f(x) - f(a) = \lim_{x\to a^-} f(x) - f(a) = 0$, what can we conclude about $\lim_{x\to a} f(x) - f(a)$? Why?

(e) Since we know $\lim_{x\to a} f(x) - f(a)$, why can we find $\lim_{x\to a} f(x)$? Hint: This is fairly easy. Use property 1 and property 7 of limits on pages 111–112 of the text.

6. In problem #37, you are asked to give an example where $\lim_{x\to a}(f(x) + g(x))$ exists, but $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x)$ does not exist. Why does property 1, page 111 not apply to show the sum of the limits is the limit of the sums?

7. Can you find an example where $\lim_{x\to 0}(f(x) + g(x))$ exists and $\lim_{x\to 0} f(x)$ exists, but $\lim_{x\to 0} g(x)$ does not exist?