Recitation 7 | MA113:004–006
17 September 1998 | Fall 1998

Reminders:
1. The project on hypocycloids is due on Friday, 9/18/98. 2. The first exam will be in CB114 from 7:30–9:30pm, Tuesday 22 September. 3. There will be a quiz on Friday over section 1.6. 4. The third graded homework assignment will be due on Tuesday 22 September in recitation. The assignment is §1.6, #58, p. 93 #16 and Appendix C, # 24.
Below is a selection of problems related to mathematical induction and the trigonometric functions. These problems will not be collected or graded. However, you should understand how to work each of these problems. You should begin working on these problems in groups in recitation. You will probably want to finish these problems outside of class. If you have questions, please ask your TA or instructor. If you find a problem difficult, consider working similar problems from the text for additional practice.

1. Use mathematical induction to prove that for each $n = 1, 2, \ldots$,
$$\sum_{k=1}^{n} k = n(n+1)/2.$$

2. (a) Find a simple formula for
$$\sum_{k=1}^{n} (k+1)^2 - k^2 = 2^2 - 1 + (3^2 - 2^2) + \ldots + n^2 - (n-1)^2 + (n+1)^2 - n^2.$$
   (b) Using your answer to part a), find another proof of the formula in the previous problem. To do this you should simplify each summand on the left.

3. Use mathematical induction to prove that
$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6.$$

4. Use mathematical induction to prove the following simple form of the binomial theorem: For $n = 1, 2, \ldots$, we have
$$(a+h)^n = a^n + na^{n-1}h + h^2 P_n(h)$$
where $P_n(h)$ is a polynomial in h. The coefficients of this polynomial depend on a.

5. Let $f(x) = x^2$ and compose f with itself n times to form $g_n(x) = f \circ f \circ \ldots \circ f$. Use mathematical induction to prove that

$$g_n(x) = x^{(2^n)}.$$

6. Work problems 15, 16 and 18 on page 93 of Stewart.

7. Appendix C, # 21, 23, 25, 27, 29, 30, 33, 37, 46, 48. I apologize that we have not directly discussed the material from appendix C, yet. We will do this on Friday.

Since the text only briefly mentions mathematical induction, the main ideas of mathematical induction from Wednesday's lecture appear below.
Principle of mathematical induction Suppose that P_n is a sequence of statements depending on a natural number $n = 1, 2, \ldots$. If we can show that:

- P_1 is true
- If P_n is true, then P_{n+1} is true.

Then, we can conclude that all the statements P_n are true.
The reason this works is that if we know P_1 is true, then the second step allows us to conclude P_2 is true. Now that we know P_2 is true, then the second step allows us to conclud P_3 is true. If we repeat this $n-1$ times, we know that P_n is true.
This principle is useful because it allows us to prove an infinite number of statements are true in just two easy steps!
Below are several examples to illustrate how to use this principle. Also, refer to pages 88 and 91 of the text.
Example Show that for $n = 1, 2, 3, \ldots$, the number $4^n - 1$ is a multiple of 3.
Solution Step 1. We need to show this is true when $n = 1$. This is easy since $4^1 - 1 = 4 - 1 = 3$ and 3 is divisible by 3.
Step 2. We suppose that $4^n - 1$ is a multiple of 3. This means that for some whole number N, $4^n - 1 = 3N$. Now $4^{n+1} - 1$. If we add and subtract 4, we have

$$4^{n+1} - 1 = 4^{n+1} - 4 + 4 - 1 = 4(4^n - 1) + 3.$$

Now we use our assumption that $4^n - 1$ is a multiple of 3 to replace $4^n - 1$ by $3N$ and obtain that

$$4^{n+1} - 1 = 4 \cdot 3N + 3 = 3(4N + 1).$$

Thus we have shown that $4^{n+1} - 1$ is a multiple of 3.

Example Show that for $n = 1, 2, \ldots$, we have

$$\sum_{k=1}^{n} 2k = n(n+1).$$

Solution Step 1. If $n = 1$, then $n(n+1) = 1 \cdot 2 = 2$. Also,

$$\sum_{k=1}^{1} 2k = 2.$$

Thus both sides are equal if $n = 1$.
Step 2. Now suppose that the formula is true for n and consider the sum

$$\sum_{k=1}^{n+1} 2k = \sum_{k=1}^{n} 2k + 2(n+1).$$

We use our assumption that $\sum_{k=1}^{n} 2k = n(n+1)$ to conclude that

$$\sum_{k=1}^{n+1} 2k = n(n+1) + 2(n+2).$$

Simplifying this last expression gives

$$n(n+1) + 2(n+1) = n^2 + n + 2n + 2 = n^2 + 3n + 2 = (n+2)(n+1).$$

Since $(n+2)(n+1) = (n+1+1)(n+1)$, we have shown that the formula

$$\sum_{k=1}^{n+1} 2k = (n+1+1)(n+1)$$

is true for $n+1$. This completes the proof by induction.

Finally, let me explain the use of the $\sum$ for summation. The notation

$$\sum_{k=1}^{n} f(k)$$

means to evaluate the function $f(k)$ at $k = 1, 2, \ldots, n$ and add up the results. In other words:

$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \ldots + f(n).$$

For example:

$$\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16,$$

$$\sum_{k=1}^{n} 2k - 1 = 1 + 3 + 5 + \ldots + 2n - 1,$$

and

$$\sum_{k=1}^{n} 1 = n.$$