MA114:010-012 — Calculus II

Longer assignment 3 — April 20, 2000

The goal of this project is to establish an amusing formula relating π to an infinite product, which is known as Wallis's product. (Of course this infinite product has to be defined as a limit.) The formula is:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots. \qquad (1)$$

The right-hand side of this expression can also be written as

$$\lim_{n \to \infty} \prod_{k=1}^{n} \frac{(2k)^2}{(2k-1)(2k+1)}.$$

Though we have not used the symbol $\prod$ before, you should be able to guess what it means, once I tell you that $\prod$ is related to products in the same way that $\sum$ is related to sums.

Your assignment is to carry out the steps below in order to see why (1) is true. The tools you will use are integration by parts (which is probably the most useful technique for studying integrals) and the comparison properties of integrals on page 245.

You should work in groups of 1 to 3 students. Each group will hand in one report for the group and all group members will receive the same grade. If someone puts there name on a report to which they made no contribution, this will be dealt with as plagiarism and they may fail the course. This assignment will be worth 15 points.

In your writeup, you should not just give the computations without explanation. The best papers will probably not give every detail of every step of the computation but will give a sentence or two describing what is being done and then the result of several steps. When you use results from the text (or from other sources) be sure to give a precise reference by page and equation number.

You should feel free to ask for help–tell us where you are stuck, and we will try to point you in the right direction. You are also free to consult with your fellow students. However, you should not copy directly from another paper, obviously.

1. Derive the reduction formula for cos, #34 inside the back cover of Varberg, Purcell and Rigdon. Hint: What I really wanted to do was ask you to derive the reduction formula for sin, #33. Unfortunately, this is done for you in section 8.4. You should imitate the argument used in example 6, p. 389 to establish the reduction formula for cos.

2. Use the reduction formula for sin to show that for $n = 2, 3, \ldots$,

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx.$$

3. Show that for $n = 1, 2, \ldots$

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n}{2n+1} = \prod_{k=1}^{n} \frac{2k}{2k+1}$$

and

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} = \frac{\pi}{2} \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

Hint: Before you try the general formula, you might want to try a few concrete examples. You should be able to evaluate $\int \sin^n x\, dx$ when $n = 0, 1, 2$. Use mathematical induction to establish the formulae for all n.

4. Divide the two formulae in part 3 and show that for $n = 1, 2, \ldots$,

$$\frac{\pi}{2} \frac{\int_0^{\pi/2} \sin^{2n+1} x\, dx}{\int_0^{\pi/2} \sin^{2n} x\, dx} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}.$$

5. To finish, we need to show that

$$\lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n+1} x\, dx}{\int_0^{\pi/2} \sin^{2n} x\, dx} = 1. \qquad (2)$$

To do this, we begin with the inequalities

$$\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x, \qquad n = 0, 1, 2 \ldots.$$

For which x are these true? Why? Using these inequalities and part 2, show that

$$f(n) \le \frac{\int_0^{\pi/2} \sin^{2n+1} x\, dx}{\int_0^{\pi/2} \sin^{2n} x\, dx} \le h(n)$$

where

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} h(n) = 1.$$

Now the squeeze theorem tells us that (2) and hence (1) holds. In your report, you should find explicit formulae for the functions f and h.

6. Who was Wallis and how did this product arise?

The above is adapted from the book *Calculus*, by Michael Spivak. In your copious spare time, look through this book to see a different view of what a calculus book should be.