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Below are revised versions of the homework problems. Sorry about the mistakes.

1. Chapter 6, #10 continued.

   Use the change of variables $s = t^2$ to express

   $$\int_0^1 (1-t^2)^{(n-1)/2}\, dt \tag{1}$$

   in terms of the beta function. The beta function is defined for $a > 0$ and $b > 0$ by

   $$B(a,b) = \int_0^1 (1-s)^{a-1} s^{b-1}\, ds.$$

   Use the formula

   $$B(a,b) = , (a), (b)/, (a+b)$$

   to express the integral (1) in terms of the , function.

   Derive and prove a formula for v_n, the volume of the unit ball in $\mathbf{R}^n$.

2. Assume that $f \geq 0$ on $(0,\infty)$. Consider the integral I defined by

   $$I = \int_{\{x: x_k > 0, k=1,\dots,n\}} f(x_1 + \dots + x_n) x_1^{a_1-1} \dots x_n^{a_n-1}\, dx.$$

   (a) Show that $f(x_1 + \dots + x_n)$ is measurable on $\mathbf{R}^n$. (Hint: Imitate the proof that $f(x-t)$ is measurable on $\mathbf{R}^{2n}$ in our study of convolutions.)

   (b) Use a linear change of variables $t = x_1 + \dots x_n$, and $t_i = x_i$, $i = 1, \dots, n-1$ and then a scaling to conclude that

   $$\begin{aligned} I = &\int_0^\infty f(t) t^{(\sum_{i=1}^n a_i)-1}\, dt \\ &\times \int_{\{t \in \mathbf{R}^{n-1}: t_1 + \dots t_{n-1} < 1, t_i > 0\}} (1 - t_1 - \dots - t_{n-1})^{a_n - 1} \\ &\times t_1^{a_1-1} \dots t_{n-1}^{a_{n-1}-1}\, dt_1 \dots dt_{n-1}. \end{aligned}$$

   (c) Now we want to evaluate the $n-1$ dimensional integral in the previous part. Our strategy is to choose a particular f for which we can evaluate both sides of (2) explicitly and then deduce the value of the integral. The magic f is $f(t) = e^{-t}$. Using this conclude that if $a_i > 0$, $i = 1 \dots n$, then

   $$\int_{\{t \in \mathbf{R}^{n-1}: t_1 + \dots t_{n-1} < 1, t_i > 0\}} (1 - t_1 - \dots - t_{n-1})^{a_n-1} t_1^{a_1-1} \dots t_{n-1}^{a_{n-1}-1}\, dt_1 \dots dt_{n-1}$$

   Observe that the last part of this problem establishes the formula for the beta function used above.

3. Show that if $g(x) = F(|x|)$, and $g \geq 0$, then

$$\int_{\mathbf{R}^n} g(x)\, dx = c_n \int_0^\infty F(t) t^{n-1}\, dt.$$

Hint: Write $F(|x|) = F \circ S(x_1^2 + \ldots x_n^2)$ where $S(t) = \sqrt{t}$. Consider the integral

$$\int_{\{x_i > 0, i=1\ldots n\}} F \circ S(x_1^2 + \ldots + x_n^2)\, dx_1 \ldots dx_n$$

make the change of variables $x_i^2 = t_i$ and apply the previous problem. Your proof should give the value of c_n.