1. Chapter 9 , #9.

2. Imitate the proof of theorem 9.1 to show that if

$$K \star f(x) = \int_0^\infty K(x/y)f(y)dy/y,$$

then for $1 \le p \le \infty$

$$\|K \star f\|_{L^p(dy/y)} \le \|K\|_{L^1(dy/y)} \|f\|_{L^p(dy/y)}.$$

Hints: The $\star$ product is commutative. Also, it may be useful to know that dy/y is the Haar measure for the group $(0, \infty)$ with multiplication as the group product. This implies that

$$\int_0^\infty f(ax)dx/x = \int_0^\infty f(x)dx/x$$

and

$$\int_0^\infty f(1/x)dx/x = \int_0^\infty f(x)dx/x.$$

If you wish to use these hints, you should prove them as part of your solution.

3. Let $\mathcal{H}$ be the collections of real-valued functions f such that f is absolutely continuous on $[0, 1]$, $f(0) = f(1) = 0$ and so that the norm on $\mathcal{H}$ given by

$$\|f\| = \left(\int_0^1 |f'(t)|^2 \, dt \right)^{1/2}$$

is finite.

   (a) Prove $\mathcal{H}$ is a Hilbert space over $\mathbf{R}$.

   (b) Prove that

   $$|f(x)| \le \|f\|.$$

   Hint: Since f is absolutely continuous on $[0, 1]$, a fundamental theorem of calculus holds.

   (c) Consider

   $$I(f) = \int_0^1 \frac{1}{2}(|f'(x)|^2 + p(x)f(x)^2) - f(x)F(x) \, dx$$

   where $p \in L^\infty([0, 1])$ and is nonnegative and F is in L^1.
   Prove that

   $$I(f) \ge \frac{1}{2}\|f\|^2 - \|F\|_{L^1}\|f\|.$$

   Conclude that $I(f)$ is bounded below on $\mathcal{H}$.

(d) Let f_i be a sequence in $\mathcal{H}$ for which

$$\lim_{i \to \infty} I(f_i) = \inf_{\mathcal{H}} I(f).$$

Prove $\{f_i\}$ converges in $\mathcal{H}$. (Hint: Imitate the proof that projections exist in Hilbert space.)

(e) Let u be the limit of the sequence I and show that

$$I(u) = \inf_{f \in \mathcal{H}} I(f).$$

(f) Compute

$$\left.\frac{d}{dt}\right|_{t=0} I(u + t\phi), \qquad \phi \in \mathcal{H}.$$

Why is this derivative 0? (Hint: This is easy.)

(g) Using your answer to the previous question, show that if u has two continuous derivatives, then

$$-u'' + pu = F.$$

(h) Show that if u has two continuous derivatives and satisfies

$$-u'' + pu = F,$$

then

$$\int_0^1 u'\phi' + pu\phi - F\phi \, dx = 0, \qquad \phi \in \mathcal{H}.$$