Long-Time Asymptotics for the Kadomtsev-Petviashvili I (KP I) Equation

Samir Donmazov
Joint work with Peter Perry and Jiaqi Liu.

University of Kentucky

Analysis and PDE Seminar, September 10, 2024

Content

1. Introduction: Our Theorem 1 and global well-posedness
2. Current literature on large-time asymptotics for u and u_x
3. Inverse scattering transform (IST) for the KP I equation
4. Our Theorem 2 (a detailed version of Theorem 1) and its proof

The KP I Equation

The KP I equation is a nonlinear dispersive partial differential equation in two spatial dimensions:

$$\begin{cases} (u_t + 6uu_x + u_{xxx})_x = 3u_{yy} \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1}$$

that describes nonlinear, long waves of small amplitude with weak dispersion in the transverse direction. It may be used to model waves in thin films with high surface tension.

Theorem 1: Large-Time Asymptotics for the KP I

Theorem 1 (SD, JL, PP)

Suppose that the initial data for the KP I equation lies in Z_w and is small in specific norms. Let

$$a = \frac{1}{12}\left(\frac{x}{t} - \frac{y^2}{12t^2}\right)$$

Then,

$$u(t,x,y) \underset{t\to\infty}{\sim} \begin{cases} o\left(t^{-1}\right), & a > 0, \\ \mathcal{O}\left(t^{-\frac{2}{3}}\right), & a \sim 0, \\ \mathcal{O}\left(t^{-1}\right), & a < 0. \end{cases} \tag{2}$$

Here, Z_w and the norms in which u is small will be defined later.

See Theorem 2 for more detailed asymptotics in different space-time regions.

Joint work with Peter Perry and Jiaqi Liu.

Global Well-Posedness for KP I Equation

Molinet, Saut, and Tzvetkov [4] proved the global well-posedness of the KP I equation for initial data belonging to the function space:

$$Z = \{u \in L^2(\mathbb{R}^2) : \|u\|_Z < \infty\}$$

with the norm

$$\|u\|_Z = \|u\|_{L^2(\mathbb{R}^2)} + \|u_{xxx}\|_{L^2(\mathbb{R}^2)} + \|u_y\|_{L^2(\mathbb{R}^2)} + \|u_{xy}\|_{L^2(\mathbb{R}^2)} + \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)} + \|\partial_x^{-2} u_{yy}\|_{L^2(\mathbb{R}^2)}. \tag{3}$$

Function Space for the Initial Data

For our large-time asymtptotic analysis, we define

$$\|u\|_{Z_w} = \|u\|_{L_x^{2,2}L_y^{2,3}} + \|u_x\|_{L_x^{2,1}L_y^{2,3}} + \|u_y\|_{L_x^{2,1}L_y^{2,2}} + \|u_{xx}\|_{L_x^2 L_y^{2,2}} + \|u_{xy}\|_{L_x^2 L_y^{2,2}} + \|u_{xxx}\|_{L_x^2 L_y^{2,2}} + \|\partial_x^{-1} u\|_{L_x^2 L_y^{2,1}} + \|\partial_x^{-1} u_y\|_{L_x^2 L_y^{2,1}} + \|\partial_x^{-2} u_{yy}\|_{L_x^2 L_y^2} + \left\|\partial_y {}_x^{-\frac{1}{2}} u\right\|_{L_x^2 L_y^{2,1}} + \|\partial_x^{-1} u\|_{L_x^2 L_y^2} + \|\partial_y \partial_x^2 u\|_{L_x^2 L_y^{2,1}} + \|\partial_y {}_x^2 u\|_{L_x^2 L_y^{2,1}} + \|\partial_y {}_x^3 u\|_{L_x^2 L_y^{2,1}} \tag{4}$$

where $\|f\|_{L_x^{2,p} L_y^{2,q}} := \left(\iint (1+x^2)^p (1+y^2)^q |f(x,y)|^2 \, dy \, dx\right)^{1/2}$.
Note that

$$\|u\|_Z \lesssim \|u\|_{Z_w}, \tag{5}$$

i.e., Z_w is continuously embedded in Z.

Literature Result 1: Leading Asymptotics for KP I

Manakov, Santini and Takhtajan [3] formally derived the leading asymptotics for the KP I equation using the stationary phase method as follows: As $t \to \pm\infty$,

$$u(t,x,y) = -\frac{1}{t} r_\xi(\xi,\eta)\,\mathrm{Re}\left(K(\xi,\eta)e^{16itr^3} + o(1)\right) \tag{6}$$

with small initial data, where

$$r^2 = \frac{1}{144}(\eta^2 - 12\xi) \tag{7}$$

with "slow" variables $\xi = x/t$ and $\eta = y/t$, and $K(\xi,\eta)$ is an approximation to the solution of a Gelfand-Levitan-Marchenko integral equation by stationary phase methods.

Note that the leading asymptotic in (6) **holds only in the space-time region**

$$\eta^2 - 12\xi > 0. \tag{8}$$

Literature Result 2: Large-Time Asymptotics for u_x of the KP Equation

Hayashi and Naumkin [2] prove that for small initial data with $\partial_x^{-1} u_0 \in H^7 \cap H^{5,4}$, the x-derivative of the solution to the KP equation has an asymptotic expansion of the form

$$u_x(t,x,y) = t^{-1}\left(\operatorname{Re} A(z) V\left(\kappa, \frac{y}{2\sigma t}\kappa\right) + o(1)\right)$$

where $A(z)$ is a "half derivative Airy function"

$$A(z) = \frac{\sqrt{2}}{\sqrt{3}\pi} e^{-\pi\sigma/4} \int_0^\infty \sqrt{\xi} e^{i(z\xi + \xi^3/3)} d\xi$$

with $\sigma = -1$ for the KP I equation and $\sigma = +1$ for the KP II equation, and V is an L^∞ function and $\kappa = (3t)^{-1/3}\sqrt{\max(0,-z)}$, $z = (3t)^{-1/3}\left(x + \frac{y^2}{4\sigma t}\right)$.

Literature Result 3: Large-Time Asymptotics for u_x of the KP I Equation

Harrop-Griffith, Ifrim and Tataru [1] show that the x-derivative of the solution to the KP I equation satisfies the pointwise bound

$$||u_x(t)||_{L^\infty} \lesssim \epsilon t^{-1/2} < t >^{-1/2}$$

if the initial data has a small norm

$$||u_0||_X \leq \epsilon \ll 1,$$

where

$$||u(0)||_X^2 = ||u(0)||_{L^2}^2 + ||u_{xxx}(0)||_{L^2}^2 + ||y^2 u_x(0)||_{L^2}^2 + ||(x\partial_x + y\partial_y)u(0)||_{L^2}^2$$

and X is a Galilean-invariant space.

Reconstruction Formula

A solution to the KP I equation is constructed through the Zhou's IST [5] as

$$u(t,x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\iint e^{itS_0(k,l;\xi,\eta)}\left(T^+(k,l)+T^-(k,l)\right) \times \mu^l(l,x;y,t)\,dl\,dk \qquad (9)$$

where

$$S_0(k,l;\xi,\eta) = (l-k)\xi - (l^2-k^2)\eta + 4(l^3-k^3)$$

is the phase function, $T^{\pm}(k,l)$ are scattering data and $\mu^l(l,x;y,t)$ is the solution to a nonlocal Riemann-Hilbert problem (RHP).

Time-Zero Scattering Data and Scattering Solutions

In the direct problem, time-zero scattering data is constructed through the initial data $u(x, y)$ and scattering solutions $\mu^{\pm}(k, x; y)$:

$$T^{\pm}(k, l) = -\frac{i}{\sqrt{2\pi}} H(\pm(l-k)) \int e^{i(l^2-k^2)\eta} \widetilde{u} * \widetilde{\mu}^{\pm}(l-k, \eta; k)\, d\eta \quad (10)$$

where $\widetilde{u}$ and $\widetilde{\mu}^{\pm}$ are the partial Fourier transforms of u and $\mu^{\pm}$ in the x variable, respectively, and $\mu^{\pm}$ is the solution of the equation

$$\begin{aligned} i\mu_y + \mu_{xx} + 2ik\mu_x + u(x, y)\mu &= 0 \\ \lim_{x \to \pm\infty} \mu(k, x; y) &= 1 \end{aligned} \quad (11)$$

which can be analytically continued to $\pm \operatorname{Im} k > 0$.

Integral Equations for $\mu^{\pm}$

$\widetilde{\mu}^{\pm}$ obey the integral equation

$$\widetilde{\mu}^{\pm} = \sqrt{2\pi}\delta(l) + g_u^{\pm}(\widetilde{\mu}^{\pm}) \tag{12}$$

where

$$g_u^{\pm}(f)(l;y) = \frac{i}{\sqrt{2\pi}} \int_{\pm l \cdot \infty}^{y} e^{-il(l+2k)(y-\eta)} (\widetilde{u} * f)(l;\eta)\, d\eta. \tag{13}$$

Let

$$\widetilde{\mu}_{\#}^{\pm}(k,l;y) = \widetilde{\mu}^{\pm}(k,l;y) - \sqrt{2\pi}\delta(l)$$

Then, (12) can be written as an integral equation for $\widehat{\mu}^{\pm}$:

$$\widetilde{\mu}_{\#}^{\pm} = g_u^{\pm}(\sqrt{2\pi}\delta) + g_u^{\pm}(\widetilde{\mu}_{\#}^{\pm}) \tag{14}$$

Existence of $\mu^{\pm}$ and Small Initial Data

The resolvent operator $(I - g_u^{\pm})^{-1}$ is bounded from $L_y^\infty L_{l,k}^2$ to itself and from $L_{k,y}^\infty L_l^1$ to itself such that

$$\|(I - g_u^{\pm})^{-1}\|_{L_y^\infty L_{l,k}^2} \leq \sum_{n=0}^{\infty} \left(\frac{\|\widetilde{u}\|_{L_{l,y}^1}}{\sqrt{2\pi}} \right)^n \frac{\|\widetilde{u}\|_{L_y^2 L_l^{2,-1}}}{\sqrt{\pi}}$$

$$\|(I - g_u^{\pm})^{-1}\|_{L_{k,y}^\infty L_l^1} \leq \sum_{n=0}^{\infty} \left(\frac{\|\widetilde{u}\|_{L_{l,y}^1}}{\sqrt{2\pi}} \right)^{n+1} \qquad (15)$$

where $\|f\|_{L_y^2 L_l^{2,-1}} := \left(\int |l|^{-1} |f(l,y)| \; dl \; dy \right)^{\frac{1}{2}}$. Hence, we require

$$\|\widetilde{u}\|_{L_{l,y}^1} < \sqrt{2\pi}, \quad \|\widetilde{u}\|_{L_y^2 L_l^{2,-1}} < \infty. \qquad (16)$$

With (16), the forward scattering map $\mathcal{S} : \widetilde{u} \mapsto T^{\pm}$ is continuous from $L_{l,y}^1 \cap L_y^2 L_l^{2,-1}$ to $L_{l,k}^2$.

Nonlocal RHP

The function $\mu^l(k, x; y, t)$ in the reconstruction formula (9) solves the nonlocal RHP

$$\mu^l = 1 + \mathcal{C}_T \mu^l \tag{17}$$

which is determined by the time-evolved scattering data

$$T(t, k, l) = e^{4it(l^3 - k^2)} T^{\pm}(k, l)$$

for $\mu^l(\,\cdot\,, x; y, t) - 1 \in L^2_k(\mathbb{R}^2)$.
Here

$$\mathcal{C}_T = C_+ \mathcal{T}^- + C_- \mathcal{T}^+, \tag{18}$$

$$(\mathcal{T}^{\pm} f)(k) = \int e^{itS_0(k,l;\xi,\eta)} T^{\pm}(k, l) f(l)\, dl \tag{19}$$

and $C_{\pm} : L^2_k(\mathbb{R}) \to L^2_k(\mathbb{R})$ denoting the Cauchy projectors.
The existence of a solution to the nonlocal RHP requires

$$C := \frac{\|\widetilde{u}\|_{L^1_{l,y}}}{\sqrt{2\pi}} < 1, \quad \|\widetilde{u}\|_{L^2_y L^{2,-1}_l} < \frac{1 - C}{4}. \tag{20}$$

Change of Variables

Define

$$a = \frac{1}{12}\left(\xi - \frac{\eta^2}{12}\right), \tag{21}$$

so that

$$r^2 = -a,$$

where r^2 is defined in (7) with $\xi = x/t$ and $\eta = y/t$ as before. For convenience, we will make the following change of variables:

$$(k, l) \to \left(\frac{\eta}{12} + k, \frac{\eta}{12} + l\right) \tag{22}$$

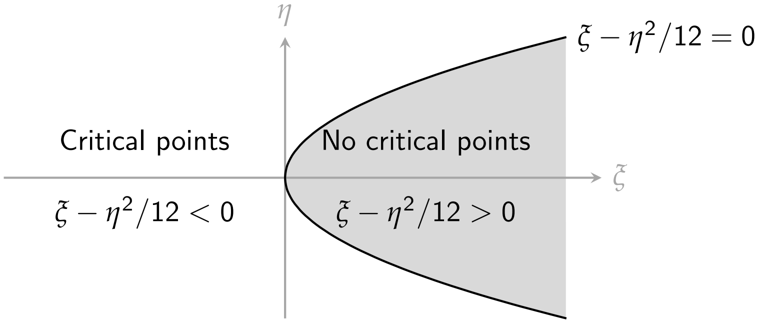
so that the phase function in shifted variables becomes

$$S(k, l; a) = 12a(l - k) + 4(l^3 - k^3) \tag{23}$$

Phase Function and Space-Time Regions

The phase function $S(k, l; a)$ in (23) with $a = (\xi - \eta^2/12)/12$ has

1. no critical points for $a > 0$,
2. a single degenerate critical point at $(0, 0)$ for $a = 0$,
3. four non-degenerate critical points, $(\pm\sqrt{-a}, \pm\sqrt{-a})$ for $a < 0$.



Reconstruction Formula Revisited

Write

$$\widetilde{T}^{\pm}(k,l) = T^{\pm}\left(k+\frac{\eta}{12}, l+\frac{\eta}{12}\right) \tag{24}$$

Let

$$A(k,l) = i(l-k)\left(\widetilde{T}^{+}(k,l)+\widetilde{T}^{-}(k,l)\right).$$

The reconstruction formula can be written as

$$u(t,x,y) = u_{loc}(t,x,y) + u_{nonloc}(t,x,y) \tag{25}$$

where

$$u_{loc}(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;a)}A(k,l)\,dk\,dl \tag{26}$$

and

$$\begin{aligned} u_{nonloc}(t,x,y) &= \frac{1}{\pi}\int e^{itS(k,l;a)}A(k,l)\left(\mu^{l}\left(l+\frac{\eta}{12},x;y,t\right)-1\right)dl\,dk \\ &+ \frac{1}{\pi}\int e^{itS(k,l;a)}\left(\widetilde{T}^{+}(k,l)+\widetilde{T}^{-}(k,l)\right)\frac{\partial\mu^{l}}{\partial x}\left(l+\frac{\eta}{12},x;y,t\right)dl\,dk. \end{aligned} \tag{27}$$

Theorem 2: Large-Time Asymptotics for KP I

Theorem 2 (SD, JL, PP)

Suppose that $u \in Z_w$, and u obeys (20). The following asymptotics hold as $t \to \infty$:

a.

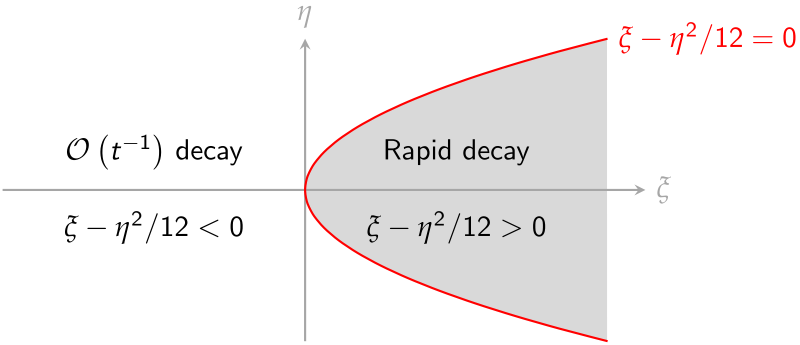
$$u_{loc}(t,x,y) \underset{t\to\infty}{\sim} \begin{cases} o\left(t^{-1}\right), & a > c > 0, \\ o\left(t^{-\frac{2}{3}}\right), & t^{\frac{2}{3}}|a| \leq c, \\ \dfrac{1}{t}\,\mathrm{Re}\left(e^{i(16tr^3-\pi/2)}\,\widetilde{T}^{+}(-r,r) \right. \\ \qquad \left. + \, e^{-i(16tr^3+i\pi/2)}\,\widetilde{T}^{+}(r,-r)\right) + o\left(t^{-1}\right), & a < -c < 0 \end{cases}$$

b.

$$u_{nonloc}(t,x,y) \underset{t\to\infty}{\sim} \begin{cases} \mathcal{O}\left(t^{-2}\right), & a > c > 0, \\ \mathcal{O}\left(t^{-\frac{2}{3}}\right), & t^{\frac{2}{3}}|a| \leq c, \\ \mathcal{O}\left(t^{-1}\right), & a < -c < 0. \end{cases}$$

 Joint work with Peter Perry and Jiaqi Liu.

Large-Time Decay Regions for the KP I Equation



Note:

$$a = \frac{1}{12}\left(\xi - \frac{\eta^2}{12}\right).$$

 Joint work with Peter Perry and Jiaqi Liu.

Local Term: No Critical Points

Proposition 1

Suppose that $u \in Z_w$ and u obeys (20). Suppose that $a > c > 0$. Then

$$u_{loc}(t,x,y) = o(t^{-1}). \tag{28}$$

Proof.

First, using the Green's identity

$$\int_\Omega e^{itS} A \, d\sigma = (it)^{-1} \left(\int_{\partial\Omega} e^{itS} A \frac{\nabla S \cdot \nu}{|\nabla S|^2} \, ds - \int_\Omega e^{itS} \nabla \cdot \left(\frac{A \nabla S}{|\nabla S|^2} \right) \, d\sigma \right), \tag{29}$$

where Ω is a domain in $\mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$, and $\nabla S \neq 0$.

Local No Critical Points: Proof of Proposition 1 (1/5)

Proof.

Let

$$S = S(k, l; a) = 12a(l - k) + 4(l^3 - k^3),$$

$$A^{\pm}(k, l) = i(l - k)\left(\widetilde{T}^{+}(k, l) + \widetilde{T}^{-}(k, l)\right),$$

$$\Omega^{\pm} = \{(k, l) : \pm(l - k) > 0\},$$

and

$$\Omega_R^{\pm} = \{(k, l) \in \Omega^{\pm} : l^2 + k^2 \leq R^2\}.$$

Then

$$u_{loc}(t, x, y) = \lim_{R \to \infty} u_{loc,R}(t, x, y)$$

Local No Critical Points: Proof of Proposition 1 (2/5)

Proof.

where

$$u_{loc,R}(t,x,y) = \frac{1}{it} \sum_{\{+,-\}} \left[\int_{\partial\Omega_R^\pm} e^{itS} A^\pm(k,l) \frac{\nabla S \cdot \nu}{|\nabla S|^2} \, ds - \int_{\Omega_R^\pm} e^{itS} \nabla \cdot \left(A^\pm(k,l) \frac{\nabla S}{|\nabla S|^2} \right) \, dl \; dk \right] \tag{30}$$

For the boundary term, it suffices to consider

$$\int_{\gamma_R^\pm} e^{itS} A^\pm(k,l) \frac{\nabla S \cdot \nu}{|\nabla S|^2} \, ds$$

where

$$\gamma_R^\pm = \{(k,l) : \pm(l-k) > 0, \; l^2 + k^2 = R^2\}.$$

□

Local No Critical Points: Proof of Proposition 1 (3/5)

Proof.

Note that

$$|\nabla S(k,l;a)| \sim (a+l^2+k^2) \tag{31}$$

$$|\Delta S(k,l;a)| \sim (a+l^2+k^2)^{\frac{1}{2}} \tag{32}$$

and we have the following estimate on the scattering data:

$$|(l-k)T^{\pm}(k,l)| \lesssim 1. \tag{33}$$

from which, we have

$$\left\|A^{\pm}\right\|_{L^{\infty}_{l,k}} \lesssim 1$$

with (31),

$$\left|\int_{\gamma_R^{\pm}} e^{itS} A^{\pm}(k,l) \frac{\nabla S \cdot \nu}{|\nabla S|^2}\, d\sigma\right| \lesssim \frac{R}{1+R^2}$$

vanish as $R \to \infty$, i.e., the boundary terms in (30) vanish.

Local No Critical Points: Proof of Proposition 1 (4/5)

Proof.

Let $\{g_n\} \subset C_0^\infty(\mathbb{R}^2)$. Then the integrals

$$I_\pm(g_n) = \int_{\Omega^\pm} e^{itS} g_n(k,l) d\sigma \tag{34}$$

can be integrated by parts N times to show that it is $\mathcal{O}(t^{-N})$. Let $g \in L^1(\mathbb{R}^2)$. Since $I_\pm : g \mapsto I_\pm(g)$ is a continuous map from $L^1(\mathbb{R}^2)$ to $\mathbb{C}$, then by the density argument, $I_\pm(g) = o(1)$.
It suffices to show that the amplitudes

$$\nabla \cdot \left(A^\pm \frac{\nabla S}{|\nabla S|^2} \right) = (\nabla A^\pm) \cdot \frac{\nabla S}{|\nabla S|^2} + A^\pm \nabla \cdot \left(\frac{\nabla S}{|\nabla S|^2} \right) \tag{35}$$

belong to $L^1(\mathbb{R}^2)$.

Local No Critical Points: Proof of Proposition 1 (5/5)

Proof.

To show this, we estimate

$$\left|\nabla\left(A^{\pm}\right)\cdot\frac{\nabla S}{|\nabla S|^{2}}\right|\lesssim\frac{|\widetilde{T}^{\pm}(k,l)|}{a+l^{2}+k^{2}}+\frac{\left|(l-k)\nabla\widetilde{T}^{\pm}(k,l)\right|}{a+l^{2}+k^{2}}\tag{36}$$

and

$$\begin{aligned}\left|A^{\pm}\nabla\cdot\left(\frac{\nabla S}{|\nabla S|^{2}}\right)\right| &\lesssim |A^{\pm}|\left(\frac{|\Delta S|}{|\nabla S|^{2}}+\frac{|\nabla S\cdot S''\cdot\nabla S|}{|\nabla S|^{4}}\right)\\ &\lesssim |A^{\pm}|\,(a+l^{2}+k^{2})^{-\frac{3}{2}}\end{aligned}\tag{37}$$

But we have the other estimates on the scattering data:

$$T^{\pm},\ (l-k)\nabla T^{\pm}\in L^{2}(\mathbb{R}^{2}).\tag{38}$$

It follows that both quantities in (36) and (37) are in $L^{1}(\mathbb{R}^{2})$. □

Local Term: Nondegenerate Critical Points

Proposition 2

Suppose that $u \in Z_w$ and u obeys (20). Suppose that $a < -c < 0$. Let $a = -r^2$. Then

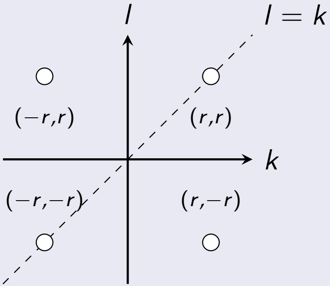
$$u_{loc}(t,x,y) \underset{t\to\infty}{\sim} \frac{1}{12t}\left(e^{-i(16tr^3-\pi/2)}\,\widetilde{T}^{+}(-r,r) + e^{i(16tr^3-\pi/2)}\,\widetilde{T}^{-}(r,-r)\right) + o(t^{-1}) \tag{39}$$

Local Nondegenerate Critical Points: Proof of Proposition 2 (1/8)

Proof.

Recall that critical points are at $(\pm r, \pm r)$. Let $\psi \in C_0^\infty$ be a cut-off function with $\psi(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\psi(s) = 0$ for $|s| \geq 1$. Define

$$\psi_a(l) = \psi\left(\frac{16(l-r)}{r}\right) + \psi\left(\frac{16(l+r)}{r}\right).$$



Local Nondegenerate Critical Points: Proof of Proposition 2 (2/8)

Proof.

Using partition of unity,

$$u_{loc}(t,x,y) = u_{loc,1}(t,x,y) + u_{loc,2}(t,x,y) \tag{40}$$

where

$$u_{loc,1}(t,x,y) = \tag{41}$$
$$= \frac{1}{\pi} \int e^{itS(k,l;a)} \psi_a(k)\psi_a(l) A(k,l)\, dl\, dk$$

and

$$u_{loc,2}(t,x,y) \tag{42}$$
$$= \frac{1}{\pi} \int e^{itS(k,l;a)} (1 - \psi_a(k)\psi_a(l)) A(k,l)\, dk\, dl$$

Local Nondegenerate Critical Points: Proof of Proposition 2 (3/8)

Proof.

Set

$$A^{\pm} = (1 - \psi_a(l)\psi_a(k)) i(l-k)\left(\widetilde{T}^{+} + \widetilde{T}^{-}\right),$$

then similar to the proof of Proposition 1, it follows that

$$u_{loc,2}(t,x,y) = o(t^{-1}). \tag{43}$$

Now, write

$$u_{loc,1} = u_{loc,1}^{+} + u_{loc,1}^{-}$$

$$u_{loc,1}^{\pm}(t,x,y) = \frac{1}{\pi}\int e^{itS(k,l;a)} H(\pm(l-k))\psi_a(k)\psi_a(l) i(l-k)\widetilde{T}^{\pm}(k,l)\,dl\,dk$$

Local Nondegenerate Critical Points: Proof of Proposition 2 (4/8)

Proof.

By an extension of Parseval's Theorem,

$$\int_{\mathbb{R}^2} f(k,l)g(k,l)\,dl\,dk = \int_{\mathbb{R}^2} \widehat{f}(-\xi_1,-\xi_2)\widehat{g}(\xi_1,\xi_2)\,d\xi_1\,d\xi_2, \tag{44}$$

where we set

$$f(k,l) = e^{itS(k,l;a)},$$
$$g(k,l) = iH(l-k)(l-k)\psi_a(k)\psi_a(l)\widetilde{T}^+(k,l).$$

With $l' = (12t)^{\frac{1}{3}} l$ and $k' = (12t)^{\frac{1}{3}} k$ scaling

$$\widehat{f}(-\xi_1,-\xi_2) = \frac{2\pi}{(12t)^{\frac{2}{3}}} \operatorname{Ai}\left((12t)^{\frac{2}{3}}\left(a - \frac{\xi_1}{12t}\right)\right) \operatorname{Ai}\left((12t)^{\frac{2}{3}}\left(a + \frac{\xi_2}{12t}\right)\right), \tag{45}$$

where

$$\operatorname{Ai}(z) = \frac{1}{2\pi}\int e^{i\left(\frac{s^3}{3}+zs\right)}\,ds \tag{46}$$

Local Nondegenerate Critical Points: Proof of Proposition 2 (5/8)

Proof.

We also have

$$\widehat{g}(\xi_1,\xi_2) = \frac{1}{2\pi}\int e^{-i(\xi_1 k+\xi_2 l)}\psi_a(k)\psi_a(l)i(l-k)H(l-k)\widetilde{T}^+(k,l)\,dl\,dk. \tag{47}$$

Let

$$A(\xi_1,\xi_2,a,t) = \mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a-\frac{\xi_1}{12t}\right)\right)\mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a+\frac{\xi_2}{12t}\right)\right)\widehat{g}(\xi_1,\xi_2), \tag{48}$$

so that

$$u^+_{loc,1}(t,x,y) = -\frac{2\pi}{(12t)^{\frac{2}{3}}}\int A(\xi_1,\xi_2,a,t)d\xi_1\ d\xi_2. \tag{49}$$

We will extract additional $t^{-1/3}$ decay from the integral in (49) to obtain the leading asymptotic of $u^+_{loc,1}(t,x,y)$ using asymptotics of the Airy function.

Local Nondegenerate Critical Points: Proof of Proposition 2 (6/8)

Proof.

Let

$$z_1 = (12t)^{\frac{2}{3}} \left(a - \frac{\xi_1}{12t} \right), \qquad z_2 = (12t)^{\frac{2}{3}} \left(a + \frac{\xi_2}{12t} \right)$$

be the arguments of the Airy functions in (48).
The leading asymptotic of the Airy function:

$$\mathrm{Ai}(-x) \underset{x \to \infty}{\sim} \frac{1}{\sqrt{\pi} x^{\frac{1}{4}}} \cos\left(\frac{2}{3} x^{\frac{3}{2}} - \frac{\pi}{4} \right) + \mathcal{O}\left(x^{-\frac{7}{4}} \right) \tag{50}$$

If $z_1 < -1$ and $z_2 < -1$, we can use the asymptotic in (50) for both Airy functions in (48):

$$\mathrm{Ai}(z_1) \underset{t \to \infty}{\sim} \frac{1}{\sqrt{\pi} r t^{\frac{1}{6}}} \cos\left(8tr^3 + \xi_1 r - \frac{\pi}{4} \right) + \mathcal{O}_r\left(t^{-\frac{7}{6}} \right) \tag{51}$$

$$\mathrm{Ai}(z_2) \underset{t \to \infty}{\sim} \frac{1}{\sqrt{\pi} r t^{\frac{1}{6}}} \cos\left(8tr^3 - \xi_2 r - \frac{\pi}{4} \right) + \mathcal{O}_r\left(t^{-\frac{7}{6}} \right) \tag{52}$$

Local Nondegenerate Critical Points: Proposition 2 (7/8)

Proof.

Let

$$\xi_1(t) = 12t(a + (12t)^{-\frac{2}{3}}), \qquad \xi_2(t) = -12t(a + (12t)^{-\frac{2}{3}}). \tag{53}$$

Write

$$u^+_{loc,1}(t,x,y) = I(t) + I^c(t) \tag{54}$$

where

$$I(t) = -\frac{2\pi}{(12t)^{\frac{2}{3}}} \int_{\xi_1 > \xi_1(t), \xi_2 < \xi_2(t))} A(\xi_1, \xi_2, a, t)\ d\xi_1\ d_2 \tag{55}$$

Note: $z_1 < -1$ implies $\xi_1 > \xi_1(t)$ and $z_2 < -1$ implies $\xi_2 < \xi_2(t)$, and

$$\begin{aligned} &4\cos(8tr^3 + \xi_1 r - \pi/4)\cos\left(8tr^3 - \xi_2 r - \pi/4\right) = \\ &e^{i(16tr^3 - \pi/2)} e^{i(\xi_1 - \xi_2)r} + e^{i(\xi_1 + \xi_2)r} \\ &+ e^{-i(\xi_1 - \xi_2)r} + e^{i(-16itr^3 + i\pi/2)} e^{-i(\xi_1 - \xi_2)r} \end{aligned} \tag{56}$$

Using asymptotics (51) with the identity (56) in (55), we recover the leading term in (39).

Local Nondegenerate Critical Points: Proof of Proposition 2 (8/8)

Proof.

On the other hand, we have the estimate

$$\left\|(1+\xi_1^2)^{\frac{1}{2}}(1+\xi_2^2)^{\frac{1}{2}}\widehat{g}\right\|_{L^2} \lesssim_r 1 \tag{57}$$

where

$$\widehat{g}(\xi_1,\xi_2) = \frac{1}{2\pi}\int e^{-i(\xi_1 k+\xi_2 l)}\psi_a(k)\psi_a(l)i(l-k)H(l-k)\widetilde{T}^+(k,l)\,dl\,dk. \tag{58}$$

The estimate (57) implies that $\widehat{g} \in L^1(\mathbb{R}^2)$ and

$$\iint\limits_{\xi_1>6tr^2} |\widehat{g}(\xi_1,\xi_2)|\,d\xi_1\,d\xi_2 \lesssim (6tr^2)^{-\frac{1}{2}}, \tag{59}$$

$$\iint\limits_{\xi_2<-6tr^2} |\widehat{g}(\xi_1,\xi_2)|\,d\xi_1\,d\xi_2 \lesssim (6tr^2)^{-\frac{1}{2}}. \tag{60}$$

It follows from (59) and (60) that $I^c(t)$ in (54):

$$I^c(t) = o(t^{-1}).$$

Local Term: Degenerate Critical Point

Proposition 3

Suppose that $u \in Z_w$, *and* u *obeys* (20). *Suppose that* $t^{\frac{2}{3}}|a| \lesssim c$. *Then*

$$u_{loc}(t,x,y) = o(t^{-\frac{2}{3}}). \tag{61}$$

Proof.

As in the proof of Proposition 2, let

$$A(\xi_1,\xi_2,a,t) = \mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a - \frac{\xi_1}{12t}\right)\right)\mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a + \frac{\xi_2}{12t}\right)\right)\widehat{g}(\xi_1,\xi_2), \tag{62}$$

where $\widehat{g} \in L^1(\mathbb{R}^2)$ with

$$\int \widehat{g}(\xi_1,\xi_2)\ d\xi_1\ d\xi_2 = 0$$

so that

$$u_{loc}(t,x,y) = -\frac{2\pi}{(12t)^{\frac{2}{3}}}\int A(\xi_1,\xi_2,a,t)d\xi_1\ d\xi_2. \tag{63}$$

Local Degenerate Critical Point: Proof of Proposition 3 (1/1)

Proof.

Note that

$$\mathrm{Ai}\left((12t)^{\frac{2}{3}}\left(a-\frac{\xi_1}{12t}\right)\right)-\mathrm{Ai}\left((12t)^{\frac{2}{3}}a\right)\underset{t\to\infty}{\sim} o_{\xi_1}(1) \tag{64}$$

Thus, by Dominated Convergence Theorem, it follows from (63) that

$$\begin{aligned} t^{\frac{2}{3}}u_{loc}(t,x,y) &= 2\pi\int \widehat{g}(\xi_1,\xi_2)\,\mathrm{Ai}\left((12t)^{\frac{2}{3}}a\right)^2 d\xi_1\,d\xi_2+o(1) \\ &= o(1) \end{aligned} \tag{65}$$

□

Large-Time Asymptotics of the Nonlocal Term

Proposition 4

Suppose that $u \in Z_w$ and u obeys (20). Then

$$|u_{nonloc}(t,x,y)| \lesssim \begin{cases} t^{-2}, & a > c > 0, \\ t^{-\frac{2}{3}}, & t^{\frac{2}{3}}|a| \leq c, \\ t^{-1}, & a < -c < 0. \end{cases} \quad (66)$$

Write

$$u_{nonloc}(t,x,y) = u_{nonloc,1}(t,x,y) + u_{nonloc,2}(t,x,y) \quad (67)$$

where

$$u_{nonloc,1}(t,x,y) = \frac{1}{\pi} \int e^{itS(k,l;a)} A(k,l) \left(\mu^l \left(l + \frac{\eta}{12}, x; y, \mathring{t} \right) - 1 \right) dl\, dk \quad (68)$$

and

$$u_{nonloc,2}(t,x,y) = \frac{1}{\pi} \int e^{itS_0(k,l;\xi,\eta)} \left(\widetilde{T}^{+}(k,l) + \widetilde{T}^{-}(k,l) \right) \times \frac{\partial \mu^l}{\partial x} \left(l + \frac{\eta}{12}, x; y, t \right) dl\, dk. \quad (69)$$

Nonlocal RHP Revisited (1/2)

Recall the nonlocal RHP:

$$\mu^l = 1 + \mathcal{C}_T \mu^l \tag{70}$$

where

$$\mathcal{C}_T = C_+ \mathcal{T}^- + C_- \mathcal{T}^+, \tag{71}$$

$$(\mathcal{T}^\pm f)(k) = \int e^{itS_0(k,l;\xi,\eta)} T^\pm(k,l) f(l)\, dl \tag{72}$$

with

$$S_0(k,l;\xi,\eta) = (l-k)\xi - (l^2-k^2)\eta + 4(l^3-k^3)$$

Let

$$\mu^l_\# = \mu^l - 1.$$

Then the nonlocal RHP becomes

$$\mu^l_\# = \mathcal{C}_T(1) + \mathcal{C}_T(\mu^l_\#). \tag{73}$$

Hence, it suffices to consider

$$(\mathcal{T}^\pm 1)(k) = \int e^{itS_0(k,l;\xi,\eta)} T^\pm(k,l)\, dl \tag{74}$$

Nonlocal RHP Revisited (2/2)

Differentiating (70) with respect to x,

$$\frac{\partial \mu^l}{\partial x} = \mathcal{C}_{\partial T/\partial x}(\mu^l) + \mathcal{C}_T\left(\frac{\partial \mu^l}{\partial x}\right) \tag{75}$$

where

$$\mathcal{C}_{\partial T/\partial x}(f) = C_+ \frac{\partial \mathcal{T}^-}{\partial x} f + C_- \frac{\partial \mathcal{T}^+}{\partial x} f \tag{76}$$

and

$$\left(\frac{\partial \mathcal{T}^\pm}{\partial x}\right)(f)(k) = \pm \int_k^{\pm\infty} e^{itS_0(k,l;\xi,\eta)} i(l-k) T^\pm(k,l) f(l)\, dl \tag{77}$$

Equation (75) can be written for $\partial \mu^l_\# / \partial x$ as

$$\frac{\partial \mu^l}{\partial x} = \mathcal{C}_{\partial T/\partial x}(I - \mathcal{C}_T)^{-1}\mathcal{C}_T(1) + \mathcal{C}_{\partial T/\partial x}(1) + \mathcal{C}_T\left(\frac{\partial \mu^l}{\partial x}\right) \tag{78}$$

Hence, it suffices to consider

$$\left(\frac{\partial \mathcal{T}^\pm}{\partial x} 1\right)(k) = \pm \int_k^{\pm\infty} e^{itS_0(k,l;\xi,\eta)} i(l-k) T^\pm(k,l)\, dl \tag{79}$$

Large-Time Asymptotics of a Solution to the Nonlocal RHP (1/2)

Lemma 3

Suppose that $u \in Z_w$, and u obeys (20). Then, the estimates following asymptotics hold as $t \to \infty$:

a

$$\left\| \mu^l - 1 \right\|_{L_l^2} \lesssim \begin{cases} t^{-1}, & a > c > 0, \\ t^{-\frac{1}{3}}, & t^{\frac{2}{3}} |a| \leq c, \\ t^{-\frac{1}{2}}, & a < -c < 0, \end{cases} \tag{80}$$

b

$$\left\| \frac{\partial \mu^l}{\partial x} \right\|_{L_l^2} \lesssim \begin{cases} t^{-1}, & a > c > 0, \\ t^{-\frac{1}{3}}, & t^{\frac{2}{3}} |a| \leq c, \\ t^{-\frac{1}{2}}, & a < -c < 0, \end{cases} \tag{81}$$

Large-Time Asymptotics of a Solution to the Nonlocal RHP (2/2)

Lemma 4

Suppose that $u \in Z_w$, and u obeys (20). Then, the estimates following asymptotics hold as $t \to \infty$:

a

$$\left\|\mathcal{T}^{\pm}(1)\right\|_{L^2_l} \lesssim \begin{cases} t^{-1}, & a > c > 0, \\ t^{-\frac{1}{3}}, & t^{\frac{2}{3}}|a| \leq c, \\ t^{-\frac{1}{2}}, & a < -c < 0, \end{cases} \tag{82}$$

b

$$\left\|\frac{\partial \mathcal{T}^{\pm}}{\partial x}(1)\right\|_{L^2_l} \lesssim \begin{cases} t^{-1}, & a > c > 0, \\ t^{-\frac{1}{3}}, & t^{\frac{2}{3}}|a| \leq c, \\ t^{-\frac{1}{2}}, & a < -c < 0, \end{cases} \tag{83}$$

References

- Benjamin Harrop-Griffiths, Mihaela Ifrim, and Daniel Tataru, The lifespan of small data solutions to the KP-I, Int. Math. Res. Not. IMRN (2017), no. 1, 1–28. MR 3632096
- Nakao Hayashi and Pavel I. Naumkin, Large time asymptotics for the Kadomtsev-Petviashvili equation, Comm. Math. Phys. 332 (2014), no. 2, 505–533. MR 3257655
- S. V. Manakov, P. M. Santini, and L. A. Takhtajan, Asymptotic behavior of the solutions of the Kadomtsev- Pyatviashvili equation (two-dimensional Korteweg - de Vries equation), Phys. Lett. A 75 (1979/80), no. 6, 451–454. MR 593107
- L. Molinet, J. C. Saut, and N. Tzvetkov, *Global well-posedness for the KP-I equation*, Math. Ann. 324 (2002), no. 2, 255–275. MR 1933858
- Xin Zhou, *Inverse scattering transform for the time dependent Schrödinger equation with applications to the KPI equation*, Commun. Math. Phys. **128** (1990), 551–564