PT-Symmetric Hamiltonians

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Parity and Time Reversal Operators

1. Parity

Let

$$\mathcal{P}(N) = \{P \in O(N) | det(P) = -1\} \quad (1)$$

be subgroup of $O(N)$ with $det = -1$, where $P^2 = 1$ for any $P \in \mathcal{P}$, i.e. $P^\dagger = P^{-1} = P$. Since P has ± 1 eigenvalues let

$$P_0 = diag\{1, 1, ..., 1, -1, -1, ..., -1\} \quad (2)$$

be the diagonal form of P with n_+ number of $+1$ and n_- number of -1 eigenvalues, where $n_+ + n_- = N$ and $(1)^{n_+}(-1)^{n_-} = -1$ (i.e. $det = -1$). Then we can write P in general as

$$P = RP_0R^{-1} \quad (3)$$

where R is N-dimensional rotation matrix having $N(N-1)/2$ free parameters.
Using

$$N(N-1)/2 - n_+(n_+ - 1)/2 - n_-(n_- - 1) \quad (4)$$

one can show that P has

$$N^2/4 - (1 - (-1)^N)/8 \quad (5)$$

free parameters.

For $N = 2$,

$$P = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \tag{6}$$

having one free parameter.

2. Time Reversal

Let

$$T = UK \tag{7}$$

where U is an unitary operator and K is c.c. operator with $K^2 = 1$.

We know that

$$T^2 = \begin{cases} 1, & \text{integer spin} \\ \text{-}1, & \text{half-integer spin} \end{cases} \tag{8}$$

PT-Symmetric Hamiltonians

1. Finite-Dimensional Matrix Representation of PT-Symmetric Hamiltonians

By definition, PT-symmetric Hamiltonians satisfy

$$[H, PT] = 0 \tag{9}$$

Let's consider $T^2 = 1$, and take $U = 1$. Also notice that $[T, P] = 0$. Then (9) implies

$$P_0 H_0^* = H_0 P_0 \tag{10}$$

where $H = RH_0R^{-1}$ with $H \in M_N(\mathbb{C})$.

For $N = 2$, let

$$H_0 = \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix}, P_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11}$$

where $a_i, b_i \in \Re$ for $i = 1, 2, 3, 4$

Solving (10) for (11) gives

$$H_0 = \begin{pmatrix} a_1 & ib_2 \\ ib_3 & a_4 \end{pmatrix} \tag{12}$$

Let H be symmetric to have orthogonal eigenvectors. Then

$$H_0 = \begin{pmatrix} a & ib \\ ib & c \end{pmatrix} \tag{13}$$

for any $a, b, c \in \Re$.

If (13) is generalized to any N,

$$H_0 = \begin{pmatrix} A & iB \\ iB^T & C \end{pmatrix} \tag{14}$$

having the 2×2 block form, where A is a real symmetric $n_+ \times n_+$ matrix, C is a real symmetric $n_- \times n_-$, and B is a real $n_+ \times n_-$ matrix.

Notice that H_0 in (14) has

$$N(N+1)/2 \tag{15}$$

free parameters. Then in accordance with (3), the number of free parameters in H is equal to the sum of the number of free parameters in H_0 (15) and the number (5) of free parameters in P, which is

$$3N^2/4 + N/2 - (1 - (-1)^N)/8 \tag{16}$$

Let's also find the most general 2×2 PT-symmetric Hamiltonian, which has four free parameters according to (16). Using the explicit form of the rotation matrix $R \in SO(2)$ and H_0 in (13), we find H as

$$H = \begin{pmatrix} a + b\cos\phi - ic\sin\phi & ic\cos\phi + b\sin\phi \\ ic\cos\phi + b\sin\phi & a - b\cos\phi + ic\sin\phi \end{pmatrix} \tag{17}$$

Note that a, b, c, ϕ (renamed free parameters) in (17) are different from those used above.

2. Algebraic Properties of PT-Symmetric Hamiltonians

Proposition 1.

Let

$$\mathcal{H}(N) = \{H \in M_N(\mathbb{C}) | [H, PT]\} \tag{18}$$

For a fixed element U in $\mathcal{H}(N)$, define a function $\phi_U : \mathcal{H}(N) \to \mathcal{H}(N)$ by $\phi_U(H) = UHU^{-1}$. Show that ϕ_U is an automorphism of $\mathcal{H}(N)$ (inner automorphism induced by U), i.e. $\phi_U \in Aut(\mathcal{H}(N))$ such that $Inn(\mathcal{H}(N)) = \{\phi_U | U \in \mathcal{H}(N)\}$ where $Inn(\mathcal{H}(N)) \leq Aut(\mathcal{H}(N))$.

Proof: Suppose $UHU^{-1} \in \mathcal{H}(N)$. We want to show that ϕ_U is isomorphism of $\mathcal{H}(N)$ to itself, i.e. $\phi_U \in Aut(\mathcal{H}(N))$.First, we show that ϕ_U is homomorphism. For any $H_1, H_2 \in \mathcal{H}(N)$, $\phi_U(H_1H_2) = UH_1H_2U^{-1} = UH_1U^{-1}UH_2U^{-1} = \phi_U(H_1)\phi_U(H_2)$. Next, we show that ϕ_U is one-to-one. Notice that $\phi_U(H_1) = \phi_U(H2) \Rightarrow UH_1U^{-1} = UH_2U^{-1} \Rightarrow U^{-1}(UH_1U^{-1})U = U^{-1}(UH_2U^{-1})U \Rightarrow H_1 = H_2$. Then we show that ϕ_U is onto. We know that $U^{=1}HU \in \mathcal{H}(N)$ for every U and H in $\mathcal{H}(N)$ since $\mathcal{H}(N)$ is a group. Now, for every H in $\mathcal{H}(N)$, notice that the image of $U^{-1}HU$ is H, $\phi_U(U^{-1}HU) = U(U^{-1}HU)U^{-1} = (UU^{-1})H(UU^{-1}) = eHe = H$. Therefore, ϕ_U is an automorphism of $\mathcal{H}(N)$.

Notice that $\phi_{PT} = id$, identity map in $Aut(\mathcal{H}(N))$ since $\phi_{PT}(H) = PTH(PT)^{-1} = H$ for every H in $\mathcal{H}(N)$ and PT in $\mathcal{H}(N)$.

Proposition 2.

Show that if $H \in \mathcal{H}(N)$, then $Char(H)(E) \in \Re[E]$, where E are energy eigenvalues of H, i.e. if H is a PT-symmetric Hamiltonian, then coefficients of its characteristic polynomial are real.

Proof 1: Note that $det(H - EI) = det(PTHT^{-1}P^{-1} - EI) = det(THT^{-1} - EI)$. If we take T in (7) where $T^{-1} = KU^{-1}$, we get $det(H - EI) = det(H^* - EI)$. Thus H and H^* have the same set of eigenvalues, then H has a real characteristic polynomial. This conclusion is valid for any choice of linear operators P and U and antilinear operator T.

Proof 2: Note that the characteristic equation of H can written as $\sum_n a_n E^n = 0$. We know that every square matrix satisfies its own characteristic equation (Cayley-Hamilton Theorem), i.e. $\sum_n a_n H^n = 0$. If $[H, \kappa] = 0$ for *any* antilinear operator κ, then H also obeys $\sum_n a_n^* H^n = 0$. Thus the characteristic polynomial of H is real.

Proposition 3.

Let H_1 and H_2 be similar matrices in $\mathcal{H}(N)$ such that $H_2 = UH_1U^{-1}$ for some inevitable matrix U. Show that H_1 and H_2 have the same characteristic polynomial.

Proof: The characteristic polynomial of H_2 is $det(UH_1U^{-1} - E_1I) = det(UH_1U^{-1} - UE_1IU^{-1}) = det(U(H_1 - E_1I)U^{-1}) = det(U)det(H_1 - E_1I)det(U^{-1}) = det(U)det(U^{-1})det(H_1 - E_1I) = det(UU^{-1}(H_1 - E_1I)) = det(H_1 - E_1I)$.