Solving KdV by using Inverse Scattering Transform

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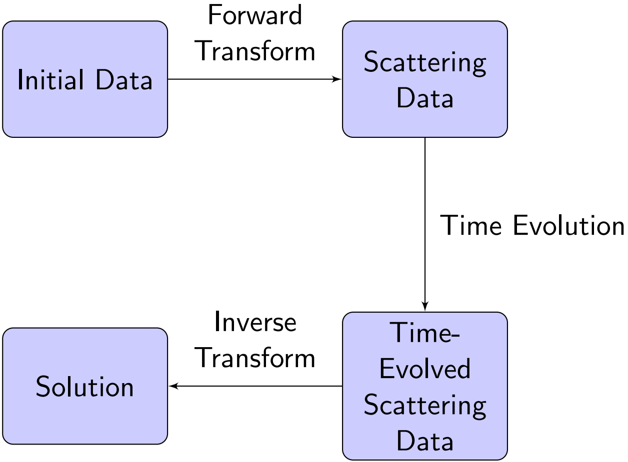
Schematic Description of Solving KdV by using IST



Associated Schrödinger Equation

The KdV equation is given by

$$q_t - 6qq_x + q_{xxx} = 0 \tag{1}$$

which is related to the modified KdV equation (MKdV)

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{2}$$

by

$$q_t - 6qq_x + q_{xxx} = (2v - i\frac{\partial}{\partial x})(v_t + 6v^2 v_x + v_{xxx})$$

where

$$q = v^2 - iv_x \tag{3}$$

is Miura transformation

So, if $v(x,t)$ is a solution of (2), then $q(x,t)$ is a solution of (1).

Since (3) is a Riccati equation, then the Miura transformation can be linearized by

$$v(x,t) = -i\frac{\psi_x}{\psi}$$

Associated Schrödinger Equation (ct'd)

which gives us the Schrödinger equation with zero energy

$$q = -\frac{\psi_{xx}}{\psi}$$

Since the KdV equation is Galilean invariant,

$$\psi_{xx} + (\lambda + q(x,t))\psi = 0$$

Gardner, Greene, Kurskal and Miura (GGKM) later discovered that the Schrödinger equation can be used to to integrate the KdV equation.

Jost Solutions of the Schrödinger Equation

Consider the KdV equation,

$$q_t - 6qq_x + q_{xxx} = 0$$

and the associated time-independent Schrödinger equation,

$$\psi_{xx} - (q - \lambda)\psi = 0$$

where $q = q(x)$ is a real potential in

$$L^1_2 = \{p(x) \mid \int_{-\infty}^{\infty} (1 + |x|^2)|p(x)|dx < \infty\}$$

Let $\psi_1(x, k)$ and $\psi_2(x, k)$,$k \in \mathbb{R} \setminus \{0\}$ be the solutions of $H\psi_j = k^2\psi_j$, $j = 1, 2$, where $\lambda = k^2$.

Jost Solutions of the Schrödinger Equation (ct'd)

Asymptotic to the solutions are

$$\psi_1(x,k) \sim e^{ikx} \quad as \quad x \to \infty, \quad \psi_2(x,k) \sim e^{-ikx} \quad as \quad x \to -\infty$$

Note that as $x \to \pm\infty$, respectively, $\psi_1(x,k)$ and $\psi_2(x,k)$ are asymptotic to sums of exponentials

$$\psi_1(x,k) \sim \frac{1}{T_2(x)} e^{ikx} + \frac{R_2(k)}{T_2(k)} e^{-ikx}, \quad x \to -\infty$$
$$\psi_2(x,k) \sim \frac{1}{T_1(x)} e^{-ikx} + \frac{R_1(k)}{T_1(k)} e^{ikx}, \quad x \to -\infty$$

where $T_2(k)\psi_1(x,k)$ describes a plane wave e^{ikx} coming from $-\infty$ which transmits $T_2 e^{ikx}$ to ∞ and reflects $R_2 e^{-ikx}$ to $-\infty$. Similarly, $T_1(k)\psi_2(x,k)$ describes scattering from ∞.

Normalized Jost Solutions

Normalized Jost solutions are defined as

$$m_1(x,k) = e^{-ikx}\psi_1(x,k), \quad m_2(x,k) = e^{ikx}\psi_2(x,k)$$

Then the time-independent Schrödinger equation becomes

$$\begin{aligned} m_1'' + 2ikm_1' &= qm_1, \\ m_2'' - 2ikm_2' &= qm_2 \end{aligned} \tag{1}$$

with $m_1 - 1 \to 0$ as $x \to \infty$ and $m_2 - 1 \to 0$ as $-\infty$.

Converting (1) into an integral equation and solving by Volterra series,

$$\begin{aligned} m_1(x,k) &= 1 + \int_0^{\infty} e^{2iky} B_1(x,y)dy, \\ m_2(x,k) &= 1 + \int_{-\infty}^{0} e^{-2iky} B_2(x,y)dy \end{aligned}$$

These representations imply that m_1, m_2 extend to $\mathbb{C}^+$, so $m_1 - 1 \in H^{2+}$, $m_2 - 1 \in H^{2+}$.

Hardy Space

H^{2+} denotes the Hardy space of functions $h(k)$ analytic in $\Im(k) > 0$ with

$$\sup_{b>0} \int_{-\infty}^{\infty} |h(a+ib)|^2 da < \infty$$

Boundary values for $h(k) \in H^{2+}$ is $h(a) = \lim_{\varepsilon \to 0} h(a+i\varepsilon) \in L^2(-\infty, \infty)$

Note that we us the following FT and IFT convention:

$$\mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} e^{2iky} f(y) dy$$
$$\mathcal{F}^{-1}\{f\}(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2iky} f(k) dk$$

Hardy Space (ct'd)

Equivalently, we have the following description,

$$H^{2+} = \{h(k) \in L^2(-\infty, \infty) \mid \operatorname{supp} \mathcal{F}^{-1}\{h\} \subset (0, \infty)\}$$

Similarly, H^{2-} denotes the Hardy space of functions analytic in $\Im(k) < 0$ and

$$H^{2-} = \{h(k) \in L^2(-\infty, \infty) \mid \operatorname{supp} \mathcal{F}^{-1}\{h\} \subset (-\infty, 0)\}$$

$h^+ = \mathcal{F}\{1_{(0,\infty)}\mathcal{F}^{-1}\{h\}\}$ and $h^- = \mathcal{F}\{1_{(-\infty,0)}\hat{h}\}$ are operators projecting L^2 onto H^{2+} and H^{2-}, respectively. Thus,

$$L^2 \cong H^{2+} \oplus H^{2-}$$

Scattering Coefficients

Let $m_1(x,k)$ and $m_2(x,k)$ be normalized Jost solutons such that $\psi_1(x,k) = e^{ikx} m_1(x,k)$ and $\psi_2(x,k) = e^{-ikx} m_2(x,k)$ solve the Schrödinger Equation

$$-\psi_j^{''} + q\psi_j = k^2 \psi_j, \ j = 1, 2,$$

with $\psi_1 \sim e^{ikx}$ as $x \to \infty$ and $\psi_2 \sim e^{-ikx}$ as $x \to -\infty$.

Note that for real $k \neq 0$, $\psi_1(x,k)$ and $\psi_1(x,-k)$ are two linearly independent solutions since the Wronskian

$$\begin{aligned}
[\psi_1(x,k), \psi_1(x,-k)] &= \psi_1^{'}(x,k)\psi_1(x,-k) - \psi_1(x,k)\psi_1^{'}(x,-k) = const. \\
&= \lim_{x\to\infty} \Big((ik)e^{ikx}e^{-ikx} - e^{ikx}(-ik)e^{-ikx} + o(1) \Big) \\
&= 2ik \neq 0
\end{aligned}$$

Similary, $[\psi_2(x,k), \psi_2(x,-k)] = -2ik \neq 0$.

Scattering Coefficients (ct'd)

So, there are unique functions $T_1(k)$, $T_2(k)$, $R_1(k)$, $R_2(k)$, called transmission and reflection coefficients, such that, for real $k \neq 0$

$$\psi_2(x,k) = \frac{R_1(k)}{T_1(k)}\psi_1(x,k) + \frac{1}{T_1(k)}\psi_1(x,-k)$$

$$\psi_1(x,k) = \frac{R_2(k)}{T_2(k)}\psi_2(x,k) + \frac{1}{T_2(k)}\psi_2(x,-k)$$

In terms of m_1 and m_2,

$$T_1(k)m_2(x,k) = R_1(k)e^{2ikx}m_1(x,k) + m_1(x,-k)$$

$$T_2(k)m_1(x,k) = R_2(k)e^{-2ikx}m_2(x,k) + m_2(x,-k)$$

Now, for real $k \neq 0$, define the scattering matrix as

$$S(k) = \begin{pmatrix} T_1(k) & R_2(k) \\ R_1(k) & T_2(k) \end{pmatrix}$$

Scattering Coefficients (ct'd)

Then,

$$\frac{1}{T_1(k)} = \frac{1}{2ik}\left[\psi_1(x,k), \psi_2(x,k)\right] = \frac{1}{T_2(k)},$$

$$\frac{R_1(k)}{T_1(k)} = \frac{1}{2ik}\left[\psi_2(x,k), \psi_1(x,-k)\right],$$

$$\frac{R_2(k)}{T_2(k)} = \frac{1}{2ik}\left[\psi_2(x,-k), \psi_1(x,k)\right],$$

Thus,

$$T_1(k) = T_2(k) = T(k), \tag{1}$$

$$R_1(k)\,T_2(-k) + R_2(-k)\,T_1(k) = 0 \tag{2}$$

with

$$\overline{T(k)} = T(-k),\ \overline{R_1(k)} = R_1(-k),\ \overline{R_2(k)} = R_2(-k) \tag{3}$$

Scattering Coefficients (ct'd)

From (1),(2),(3), we obtain

$$|T(k)|^2 + |R_1(k)|^2 = 1 = |T(k)|^2 + |R_2(k)|^2$$

Hence, S is a unitary matrix for each real $k \neq 0$.

Bound States: Norming Constants

Let H be the self-adjoint Schrödinger operator $-d^2/dx^2 + q(x)$ with a real potential $q \in L^1_2$. Then H has finite number of bound states $-\kappa_n^2 < \cdots < -\kappa_1^2$ associated with eigenfunctions $f(x, i\kappa_j)$, $j = 1, \cdots, n$. Also, norming constants are defined by

$$c_j = \left(\int_{-\infty}^{\infty} \psi^2(x, i\kappa_j) dx \right)^{-1}$$

Hence, we obtain the scattering data $\{\kappa_j, c_j, R(k)\}$, which uniquely determines the potential $u(x, t)$ for each fixed t.

Simultaneous Solutions of KdV and Schrödinger equations

Consider the associated Schrödinger equation

$$\psi_{xx} - (q - \lambda)\psi = 0 \tag{1}$$

where $q(x,t)$ is a solution of the KdV equation

$$q_t - 6qq_x + q_{xxx} = 0 \tag{2}$$

so that $\psi(x,t)$ and $\lambda(t)$ depend on parameter t. Solving (1) for q, then substituting into (2),

$$\lambda_t \psi^2 + [\psi Q_x - \psi_x Q]_x = 0 \tag{3}$$

where

$$Q = \psi_t + \psi_{xxx} - 3(q + \lambda)\psi_x$$

Simultaneous Solutions of KdV and Schrödinger Equations (ct'd)

Assume ψ vanishes as $|x| \to \infty$. Then, one can check that the second term of (3), $[\psi Q_x - \psi_x Q]_x$, vanishes on integration over $(-\infty, \infty)$. Thus, $\lambda_t = 0$. Then, (3) becomes

$$\left[\frac{Q}{\psi}\psi^2\right]_x = 0 \tag{4}$$

Integrating (4) twice,

$$\psi_t + \psi_{xxx} - 3(q + \lambda)\psi_x = C\psi + D\phi \tag{5}$$

where

$$\phi = \psi \int_{-\infty}^{x} \frac{dx'}{\psi^2}$$

Time-Evolution of Norming Constants

Now, consider time-independent discrete eigenvalues $\lambda_j < 0$, $j = 1, 2, ..., n$, where corresponding eigenfunctions ψ_j satisfy (5) .

Note that $D = 0$ since ψ_j vanishes as $|x| \to \infty$. Also, $C = 0$ since ψ_j are assumed to be orthonormal.

So, substituting $\psi_j \approx c_j(t)e^{-\kappa_n x}$ as $x \to \infty$ with $\kappa_j = (-\lambda_j)^{1/2}$ into (5),

$$c_j(t) = c_j(0)e^{4\kappa_j^3 x}$$

Time-Evolution of Scattering Coefficients

For $\lambda = k^2 > 0$, solutions of (1) at large $|x|$ are

$$\psi \approx e^{-ikx} + Re^{ikx} \quad as \quad x \to \infty \tag{6}$$

$$\psi \approx Te^{-ikx} \quad as \quad x \to -\infty \tag{7}$$

where $T(k,t)$ and $R(k,t)$ are transmission and reflection coefficients, respectively. Note that $|T|^2 + |R|^2 = 1$.

Assuming that λ is constant, then substituting (6) and (7) into (5), we obtain $D = 0$ and $C = 4ik^3$, and

$$R(k,t) = R(k,0)e^{8ik^3 t}$$

$$T(k,t) = T(k,0)$$

Thus, we obtain the time-evolved scattering data,

$$\{\kappa_j, c_j(t), R(k,t)\}$$

Derivation of Gel'fand-Levitan-Marchenko Equation

Consider the following equation

$$T(k)m_2(x,k) = m_1(x,-k) + R_1(k)e^{2ikx}m_1(x,k) \tag{1}$$

Subtracting 1 from both sides, let us rewrite the LHS of (1) as

$$T(k)(m_2(x,k)-1) + T(k) - 1 \tag{2}$$

where $T(k)$ is bounded and analytic in $\mathbb{C}^+$, $m_2(x,k) - 1 \in H^{2+}$ and $T(k) - 1 \in H^{2+}$

Also, rewrite the RHS of (1) as

$$(m_1(x,-k)-1) + R_1(k)e^{2ikx} + R_1(k)e^{2ikx}(m_1(x,k)-1) \tag{3}$$

Derivation of Gel'fand-Levitan-Marchenko Equation (ct'd)

Now, take the inverse Fourier of the first term in (3) (RHS)

$$\begin{aligned}\mathcal{F}^{-1}\{m_1(x,-k)-1\} &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2iky}(m_1(x,-k)-1)dk \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2iky}\int_0^{\infty} e^{-2iky'}B_1(x,y')dy'dk \\ &= B_1(x,y),\ \ y>0\end{aligned} \tag{1}$$

IFT of second term in (3) gives,

$$\mathcal{F}^{-1}\{R_1 e^{2ikx}\} = F_1(x+y),\ \ y>0 \tag{2}$$

Derivation of Gel'fand-Levitan-Marchenko Equation (ct'd)

Third term ...

$$\mathcal{F}^{-1}\{R_1(k)e^{2ikx}m_1(x,k)-1\} = \int_{-\infty}^{\infty} F_1(x+y+t)B_1(x,t), \;\; y>0$$

and the IFT of (2) (LHS) is

$$\mathcal{F}^{-1}\{T(k)m_2(x,k)-1\} = 0, \;\; y>0$$

Thus, we obtain the Gel'fand-Levitan-Marchenko Equation with no bound states

$$B(x,y)+F(x+y)+\int_0^{\infty} F(x+y+t)B(x,t)dt = 0, \;\; y>0$$

where $F(y) = \mathcal{F}^{-1}\{R\}(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(k)e^{2iky}dk$

Existence and Uniqueness

Consider the time-independent Gel'fand-Levitan-Marchenko Equation.

$$B(x,y) + \Omega(x+y) + \int_0^\infty \Omega(x+y+t)B(x,t)dt = 0, \;\; y > 0 \qquad (1)$$

where $\Omega(y) = F(y) + \sum_{j=1}^{n} c_j(t)e^{-2\kappa_j y}$ and $F(y)\frac{1}{\pi}\int_{-\infty}^{\infty} R(k)e^{2iky}dk$.

Fix x, and let $g(y) = B(x,y)$, where $g \in L^2[0,\infty)$. Then, (1) becomes,

$$\Omega(y) + g(y) + \int_0^\infty \Omega(y+t)g(t)dt = 0$$

Define $T(g)(y) = \int_0^\infty \Omega(y+t)g(t)dt$ such that

$$\Omega + (I+T)g = 0 \qquad (2)$$

Note that T is a Hilbert-Schmidt operator, i.e. $\Omega \in L^2(\mathbb{R})$, then T is compact. It can be shown that $\ker(I+T) = \{0\}$. Thus, $(I+T)^{-1}$ exists by Fredholm alternative. Then (2) has a unique solution, and the solution is

$$g = -(I+T)^{-1}\Omega$$

n-Soliton Solution

Consider the Gel'fand-Levitan-Marchenko.

$$B(x,y,t)+\Omega(x+y,t)+\int_x^\infty \Omega(z+y,t)B(x,z,t)dz=0,\ \ x<y<\infty \quad (1)$$

where

$$\Omega(y,t)=F(y,t)+\sum_{j=1}^{n} c_j(t)e^{-\kappa_j y}$$

$$F(y,t)=\hat{R}(y,t)=\frac{1}{2\pi}\int_0^\infty R(k,t)e^{iky}dk$$

Recover the time-dependent solution of the KdV equation by using

$$q(x,t)=-2\frac{\partial B(x,x,t)}{\partial x} \quad (2)$$

Set $R(k,t)=0$ for the n-soliton solution. Let

$$X(x):=\begin{bmatrix} e^{-\kappa_1 x} & e^{-\kappa_2 x} & \cdots & e^{-\kappa_n x}\end{bmatrix},\ \ Y(y,t):=\begin{bmatrix} c_1(t)e^{-\kappa_1 y} \\ c_2(t)e^{-\kappa_2 y} \\ \vdots \\ c_n(t)e^{-\kappa_n y}\end{bmatrix}$$

n-Soliton Solution (ct'd)

We obtain,

$$\Omega(x+y,t) = X(x)Y(y,t)$$

Assume the solution of the Gel'fand-Levitan-Marchenko Equation has the form $B(x,y,t) = H(x,t)Y(y,t)$ where $H(x,t)$ is a row vector. Substituting into (1),

$$H(x,t)Y(y,t) + X(x)Y(y,t) + \int_x^\infty H(x,t)Y(z,t)X(x)Y(y,t)dz = 0$$

Then, we obtain,

$$H(x,t) = -X(x)\Gamma(x,t)^{-1}$$

where

$$\Gamma(x,t) = I + \int_x^\infty Y(z,t)X(z)dz \tag{3}$$

So,

$$B(x,y,t) = -X(x)\Gamma(x,t)^{-1}Y(y,t) \tag{4}$$

n-Soliton Solution (ct'd)

Thus,

$$\Gamma_{i,j} = \delta_{ij} + \int_x^\infty c_i(t) e^{-(\kappa_i+\kappa_j)z}$$
$$= \delta_{ij} + c_i(0)\frac{e^{-(\kappa_i+\kappa_j)x+8\kappa_i^3 t}}{\kappa_i+\kappa_j}$$

Substituting (4) into (2),

$$q(x,t) = 2\frac{\partial\left[X(x)\Gamma(x,t)^{-1}Y(x,t))\right]}{\partial x} = 2tr\left(\frac{\partial\left[Y(x,t)X(x)\Gamma(x,t)^{-1}\right]}{\partial x}\right)$$

Note from (3) that

$$\frac{\partial\Gamma(x,t)}{\partial x} = -Y(x,t)X(x)$$

Therefore,

$$q(x,t) = -2tr\left(\frac{\partial\left[\frac{\partial\Gamma(x,t)}{\partial x}\Gamma(x,t)^{-1}\right]}{\partial x}\right) = -2\frac{\partial}{\partial x}\left[\frac{\frac{\partial \det\Gamma(x,t)}{\partial x}}{\det\Gamma(x,t)}\right]$$

One-Soliton Solution

Now, set $n = 1$,then,

$$\frac{\frac{\partial \det \Gamma(x,t)}{\partial x}}{\det \Gamma(x,t)} = \frac{-c_1(0)e^{-2\kappa_1 x + 8\kappa_1^3 t}}{1 + \left(c_1(0)e^{-2\kappa_1 x + 8\kappa_1^3 t}\right)/(2\kappa_1)} \quad (5)$$

After differentiating (5) with respect to x, let

$$\theta := \log\left[\frac{2\kappa_1}{c_1(0)}\right]^{1/2}$$

Then, we obtain

$$q(x,t) = -2\kappa_1^2 sech^2[\kappa_1 x - 4\kappa_1^3 t + \theta]$$

References

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2. P. Deift, E. Trubowitz, *Inverse Scattering on the Line*, 1979
3. C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Inverse Scattering on the Line*, 1967
4. A. C. Newell, *Solitons in Mathematics and Physics*, 1985

Appendix: Riccati Equation

The Riccati equation is given by

$$y^{'}(x) = a_2(x)y^2(x) + a_1(x)y(x) + a_0(x) \quad (1)$$

where $a_2(x) \neq 0$ and $a_0(x) \neq 0$.

(1) can always be reduced to a second order linear ODE whenever a_2 is nonzero and differentiable, then $v = ya_2$ satisfies a Riccati equation of the form

$$v^{'}(x) = v^2(x) + B(x)v(x) + C(x)$$

where $C = a_2a_0$ and $B = a_1 + q_2^{'}/q_2$. Now, substituting $v = -u^{'}/u$,

$$u^{''}(x) - B(x)u^{'}(x) + C(x)u(x) = 0 \quad (2)$$

where a solution of (2) is related to a solution of (1) by $y = -u^{'}/(a_2u)$.

Appendix: Integral Equation

Consider the time-independent Schrödinger Equation

$$m^{''} + 2ikm^{'} = qm$$

Or symbolically,

$$\partial_x(\partial_x + 2ik)m = qm \tag{1}$$

To invert the operator (1),we uses the Green's function

$$D_k(t-x) = \int_0^{t-x} e^{2ikt}dt = \frac{1}{2ik}(e^{2ik(t-x)} - 1)$$

For each k, $\Im(k) \geqslant 0$, we obtain the integral equation

$$m(x,k) = 1 + \int_x^\infty D_k(t-x)q(t)m(t,k)dt$$

Appendix: Volterra Series for $m(x,k)$

Consider the following iteration.

$$m_0(x) = 1$$

$$m_{n+1}(x,k) = 1 + \int_x^\infty D_k(t-x)q(t)m_n(t,k)dt$$

Thus, we obtain the Volterra Series

$$m(x,k) = 1 + \sum_{n=1}^{\infty} g_n(x,k)$$

where

$$g_n(x,k) = \int_{x\leqslant x_1 \cdots \leqslant x_n} D_k(x_1 - x)q(x_1)\cdots D_k(x_n - x_{n-1})q(x_n)dx_n \cdots dx_1$$

which gives us

$$m(x,k) = 1 + \int_x^\infty D_k(t-x)q(t)m(t,k)dt$$

Appendix: Volterra Series Convergence

We have

$$m(x,k) = 1 + \sum_{n=1}^{\infty} g_n(x,k)$$

where

$$g_n(x,k) = \int_{x\leqslant x_1\cdots\leqslant x_n} D_k(x_1 - x)q(x_1)\cdots D_k(x_n - x_{n-1})q(x_n)dx_n\cdots dx_1$$

Note that

$$|g_n(x,k)| \leqslant \int_{x\leqslant x_1\cdots\leqslant x_n} \frac{1}{|k|^n}|q(x_1)|\cdots|q(x_n)|dx_n\cdots dx_1 = \frac{1}{|k|^n}\frac{\left(\int_x^\infty |q(t)|dt\right)^n}{n!}$$

where $D_k(y) \leqslant 1/|k|$, $\Im(k) \geqslant 0$.

Let $\gamma(x) := \int_x^\infty |q(t)|dt/|k|$. Then, for $|k| > 0$,

$$\left|\sum_{n=1}^{\infty} g_n(n,k)\right| \leqslant e^{\gamma(x)} - 1 < \infty$$

Appendix: Volterra Series for $B(x, y)$

Taking the inverse Fourier transform of

$$m'' + 2ikm' = qm$$

we obtain

$$\partial^2 B/\partial x \partial y - \partial^2 B/\partial x^2 + qB = 0$$

Let

$$B(x, y) = \sum_{n=1}^{\infty} K_n(x, y)$$

where

$$K_0(x, y) = \int_{x+y}^{\infty} q(t)dt$$

$$K_{n+1}(x, y) = \int_0^y dz \int_{x+y-z}^{\infty} dt q(t) K_n(t, z)$$

Thus, we obtain

$$B(x, y) = \int_{x+y}^{\infty} q(t)dt + \int_0^y dz \int_{x+y-z}^{\infty} dt q(t) B(t, z), \quad y \geqslant 0$$

Appendix: Further Notes on $B(x, y)$

Note that $B(x, y)$ solves the wave equation

$$\partial^2 B/\partial x \partial y - \partial^2 B/\partial x^2 + qB = 0$$

with

$$-\partial B(x, 0+)/\partial x = -\partial B(x, 0+)/\partial y = q(x).$$

Then,

$$m(x, k) = 1 + \int_0^{\text{inf}} e^{2iky} B(x, y) dy$$

where m solves the time-independent Schrödinger Equation

$$m'' + 2ikm' = qm$$

Appendix: Fredholm Alternative

Theorem: Let $A \in \mathcal{K}(\mathcal{H})$, then either

1. $(A - I)^{-1}$ exists or
2. $A\psi = \psi$ has a solution $\psi \neq 0 \in \mathcal{H}$.