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This note is meant to sum up some elementary statements on PT-symmtric, finite dimensional QM by Samir Donmazov and Paul de Lange.

0.1 P and T

Let K denote c.c., then

$$T = K \cdot A \tag{1}$$

for a unitary operator A. We will for now take $A = \mathbf{1}$.
Let P be the parity operator.

$$P^2 = 1,\ [P, T] = 0 \tag{2}$$

We wish to formulate a theory of QM that does not presume Hermitian symmetry, that is we will only demand operators O acting on the Hilbert space $\mathcal{H}$ to commute with PT

$$[O, PT] = 0 \tag{3}$$

Note that

$$P = R^{-1} P_0 R,\ P_0 = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \tag{4}$$

Let (n_+, n_-) denote the number of positive and negative eigenvalues of P_0 respectively, and let from now on $\dim \mathcal{H} = 2n$. Then in principle, and choice of (n_+, n_-) yield a representation of P. Now surely, $n_\pm = 0$ means $P = \pm 1$ and the space of PT-symmetric operators becomes the space $GL(2n, \mathbb{R})$. We will be interested for now in the case $(n_+, n_-) = (n, n)$. This choice will put the most stringent conditions on an operator O to be PT-symmetric, but also its these representations that rotate to a basis wherein

$$P' = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & 0 \\ 0 & 1 & & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix} \tag{5}$$

and if $|\psi\rangle_i$, $i = -n, \ldots, n$ span $\mathcal{H}$, than P' just does $i \mapsto -i$ and in a continuum limit this will look a lot like space reflection $P : x \mapsto -x$.

At this point we will demand operators acting on $\mathcal{H}$ to be symmetric, as this ensures the eigenstates to be orthogonal, so that R in 4 is in $O(2n)$. In fact, R acts trivially on the first n^2 diagonal block of P_0 and idem for the lower part, so that really

$$P \in O(2n, \mathbf{R})/(O(n, \mathbf{R}) \times O(n, \mathbf{R}))/\mathbf{Z}_2 \tag{6}$$

where the $\mathbf{Z}_2$ is just overall multiplication by ± 1. Note that P has $2n^2+1$ parameters.

What is the inner product on $\mathcal{H}$? The first natural candidate is to take

$$\langle\psi| = (PT|\psi\rangle)^{\mathrm{T}} \tag{7}$$

But now

$$\langle\psi,\psi\rangle = \sum_{i=1}^{n} |\psi_i|^2 - \sum_{i=n+1}^{2n} |\psi_i|^2 \tag{8}$$

First of all, we see that for PT-symmetry also, a wave vector is defined modulo $U(1)$. Moreover the inner-product is not positive definite. In fact the group G of unitary transformations wrt this inner product is

$$G = O(n,n) \tag{9}$$

To resolve this issue, we introduce the operator C

$$C = \sum_i |\psi\rangle\langle\psi| \tag{10}$$

Note that this operator is not $C = 1$ with the current definitions of bra and ket. We now redefine the bra by

$$\langle\psi| = (CPT|\psi\rangle)^{\mathrm{T}} \tag{11}$$

so that with this new definition of bra, the completeness relation does hold. Also, now the metric is positive definite.

0.2 random PT

The goal is to write down the ensemble of random PT symmetry. First, for a PT symmetric matrix H, the proper invariant is now

$$\langle H|H\rangle = \mathrm{Tr}\left((CPT\cdot H)^T \cdot H\right) \tag{12}$$

and we propose to consider the ensemble

$$P(H) = \kappa e^{-\mathrm{Tr}\langle H,H\rangle_{CPT}} \prod_{i,j} dH_{ij} \tag{13}$$

We will assume that H is symmetric, to ensure the eigenstates of H are orthogonal.
First we'll consider the 2-dimensional case, and later work out higher and large-n.
The most general PT-invariant, symmetric Hamiltonian H is

$$H = R^{-1}\begin{pmatrix} a & ib \\ ib & c \end{pmatrix} R \tag{14}$$

$R \in O(2)$.

$$P(H) = \kappa' \, e^{-(a^2+2b^2+c^2)} da \, db \, dc \tag{15}$$

where we integrated out the $O(2)$ rotation angle.