THEORY OF THE NEAREST SQUARE CONTINUED FRACTION

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1. *Introduction*

THE genesis of the present investigation is a remark of the late Sir Thomas Little Heath that the Indian Cyclic Method of solving the equation $x^2 - Ny^2 = 1$ in integers due to Bhaskara in 1150, is* 'remarkably enough, the *same* as that which was rediscovered and expounded by Lagrange in 1768'. We have pointed out elsewhere† that the Indian method implies a half-regular continued fraction (*h.r.c.f.*, for brevity) with certain noteworthy properties which have not been previously investigated. If we remember that it was Lagrange who was mainly responsible for the neglect of the *h.r.c.f.* since he showed, by an example, how it would never uniformly lead to the solution of the so-called Pellian equation, we can appreciate the distance between Lagrange's simple continued fraction and the one discussed in this paper. This new continued fraction, we call, the nearest square continued fraction or Bhaskara continued fraction (B.*c.f.* for brevity), the natural sequel to Bhaskara's cyclic method. The whole theory can be developed as it were from 'scratch' with the help of the simplest mathematics known to the Hindus about the fifth century A.D.

2. *The New Continued Fraction Defined*

2·1. A quadratic surd of the form $\frac{P+\sqrt{R}}{Q}$ is usually expressed as the sum and not as the difference of an integer and a positive proper fraction. We now exploit both the representations at once. Herein lies the novelty of procedure.

Definition.—The surd $\frac{P+\sqrt{R}}{Q}$ is said to be in the standard form, if R is a non-square positive integer, and P ($\neq 0$), Q, $\frac{R-P^2}{Q}$ are integers having no common factor, while if P is zero, it is sufficient that $\frac{R}{Q}$ and Q are relatively prime integers.

* See page 285, *Diophantus of Alexandria*, by Sir T. L. Heath, Cambridge, 1910.
† See pages 602-604, *Curr. Sci.*, Vol. VI, No. 12, June 1938.

We shall first set down some elementary results.

THEOREM I. If a is the greatest integer in the standard surd $\frac{P+\sqrt{R}}{Q}$, and $\frac{P+\sqrt{R}}{Q} = a + \frac{Q'}{P'+\sqrt{R}} = a+1-\frac{Q''}{P''+\sqrt{R}}$, then $\frac{P'+\sqrt{R}}{Q'}$ and $\frac{P''+\sqrt{R}}{Q''}$ are also standard surds with the following properties :

(i) $P''-P'=Q$; $P''+P'=Q'+Q''$; $Q'-\frac{1}{2}Q \leqslant P'$ if $Q' \leqslant Q''$; $Q''+\frac{1}{2}Q \leqslant P''$ if $Q'' \leqslant Q'$.

(ii) $Q'^2+Q''^2+Q^2+2Q'Q''+2QQ'-2QQ''=4R$.

(iii) If $|Q'|$, $|Q''|$, $|Q|$ be all greater than $\sqrt{R}$, then $|P'|$, $|P''|$, $\frac{1}{2}|Q|$ are all greater than $\sqrt{2R}$ and at least one of $|Q'|$, $|Q''|$ is less than $\frac{1}{2}|Q|$; also P', P'', Q', Q'' are all numerically less than $|Q|$.

(iv) If $|Q| < 2\sqrt{R}$, then Q' and Q'' are both positive and at least one of them is less than $\sqrt{R}$; if $|Q| < \sqrt{2R}$, then one of P', P'' is positive; if $|Q| < \sqrt{R}$, P', P'' are both positive and less than $2\sqrt{R}$.

(v) If $|Q| < 2\sqrt{R}$, then $Q' \gtreqless Q''$ according as

$$\frac{Q'}{P'+\sqrt{R}} \gtreqless \frac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R-Q^2}}{2Q} ;$$

if $|Q| > 2\sqrt{R}$, then $|Q'| \gtreqless |Q''|$ according as $\frac{Q'}{P'+\sqrt{R}} \gtreqless \frac{1}{2} + \frac{\sqrt{R}}{Q}$.

Proof.—(i) and (ii) follow readily from the well-known relations:—

(1) $P'=aQ-P$; (2) $P''=(a+1)Q-P$; (3) $P'^2=R-QQ'$; (4) $P''^2=R+QQ''$. The elements of the triplet $\left(\frac{R-P^2}{Q}, P, Q\right)$ can be expressed as the sum of integral multiples of the elements of the triplet $\left(\frac{R'-P'^2}{Q'}, P', Q'\right)$ and *vice-versa*. Hence $\frac{P'+\sqrt{R}}{Q'}$ is also a standard surd, when $\frac{P+\sqrt{R}}{Q}$ is one. Similarly $\frac{P''+\sqrt{R}}{Q''}$ is also standard.

From (ii),
$$(Q'-Q''+Q)^2+4Q'Q'' = 4R = (Q'+Q''+Q)^2-4QQ'' = (Q'+Q''-Q)^2+4QQ'. \quad (5)$$

If $|Q|$, $|Q'|$, $|Q''|$ be all greater than $\sqrt{R}$, then $|QQ'|$, $|QQ''|$, $|Q'Q''| > R$ implies $Q'Q'' < 0$, $QQ'' > 0$, and $QQ' < 0$.

Hence Q, Q'' are of the same sign and different from that of Q'. (α)

Again, $\frac{Q'}{P'+\sqrt{R}}$ and $\frac{Q''}{P''+\sqrt{R}}$ are positive proper fractions, so that if Q', Q'' are of opposite signs, so also are the pairs $P'+\sqrt{R}$, $P''+\sqrt{R}$; and P', P'': and the latter are absolutely greater than $\sqrt{R}$. (β)

From (3), (4) and (α), $|P'| > \sqrt{2R}$, $|P''| > \sqrt{2R}$;

From (i) and (β), $|Q| = |P'| + |P''| > 2\sqrt{2R}$; and one of P', P'' is not absolutely greater than $\frac{1}{2}|Q|$.

From (3) and (α), $|QQ'| < P'^2$; but $|Q| > |P'|$; $\therefore$ $|Q'| < |P'| < |Q|$; similarly $|Q''| < |P''| < |Q|$.

Hence, $|Q'|$ or $|Q''|$ is less than $\frac{1}{2}|Q|$ according as $|P'|$ or $|P''|$ is not greater than $\frac{1}{2}|Q|$. This proves (iii).

If $|P'|$, $|P''|$ be both less than $\sqrt{R}$, we have from (3) and (4) Q, Q' of the same sign and different from that of Q''.

By (β), $P'+\sqrt{R}$ and $P''+\sqrt{R}$ must also be of opposite signs, which contradicts the assumption $(|P'|, |P''|) < \sqrt{R}$.

Hence, $|P'|$, $|P''|$ are never both less than $\sqrt{R}$. (γ)

If $|P'|$, $|P''|$ are both greater than $\sqrt{R}$, then $|Q| > 2\sqrt{R}$.

If $|Q| < 2\sqrt{R}$, one of $|P'|$, $|P''|$ is less and the other greater than $\sqrt{R}$, so that by (3) and (4), Q', Q'' are of the same sign.

When Q', Q'' are of the same sign, $P'+\sqrt{R}$ and $P''+\sqrt{R}$ are also of the same sign and the numerically greater of P', P'' must be positive, and so all the quantities $P'+\sqrt{R}$, $P''+\sqrt{R}$, Q', Q'' must be positive.

Therefore, $Q'Q'' < R$ by (5) and so one of Q', Q'' is less than $\sqrt{R}$.

Again, if either $Q' < Q''$, $P' < 0$, $P'' > 0$, or $Q' > Q''$, $P'' < 0$, $P' > 0$, we have $Q(Q'-Q'') = (P''-P')(Q'-Q'') < 0$, and by (ii) and (i), $Q^2 + (Q'+Q'')^2 > 4R$, and $|Q| > |Q'+Q''|$ and therefore $Q^2 > 2R$.

If $Q' = Q''$, and P', P'' be of opposite signs, then $Q^2 + (Q'+Q'')^2 = 4R$ and again $Q^2 > 2R$.

Therefore, when $|Q| < \sqrt{2R}$, we must have P' or P'' or both positive, according as $Q' < Q''$ or $Q' > Q''$ or $Q' = Q''$.

From (3), $|QQ'| = |\sqrt{R}-P'|\cdot|\sqrt{R}+P'|$. But $|Q'| < |P'+\sqrt{R}|$, so $|Q| > |\sqrt{R}-P'|$.

If $\sqrt{R} > |Q|$, then $\sqrt{R} > |\sqrt{R}-P'|$, *i.e.*, $2\sqrt{R} > P' > 0$; similarly $2\sqrt{R} > P'' > 0$.

(iv) is thus proved.

If $|Q| < 2\sqrt{R}$, we have from (ii), $(Q' + Q'')^2 \gtrless 4R - Q^2$ according as $Q(Q' - Q'') \lessgtr 0$. By (iv), $(Q', Q'') > 0$ and if $Q < 0$ and $Q' > Q''$,

we have, $(Q' + Q'')^2 > 4R - Q^2$

$$i.e., \quad \frac{Q' + Q''}{-2Q} > \sqrt{\frac{R}{Q^2} - \frac{1}{4}}$$

$$i.e., \quad \frac{2P' + Q}{-2Q} > \sqrt{\frac{R}{Q^2} - \frac{1}{4}}$$

$$i.e., \quad \frac{\sqrt{R} - P'}{Q} > \frac{1}{2} + \frac{\sqrt{R}}{Q} + \sqrt{\frac{R}{Q^2} - \frac{1}{4}}$$

$$i.e., \quad \frac{Q'}{P' + \sqrt{R}} > \frac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q}$$

The same result is obtained, when $Q > 0$ and $Q' > Q''$.

Hence, when $Q' > Q''$ and $|Q| < 2\sqrt{R}$,

$\dfrac{Q'}{P' + \sqrt{R}} > \dfrac{1}{2} + \dfrac{\sqrt{R}}{Q} - \dfrac{\sqrt{4R - Q^2}}{2Q}$. Similarly, when $Q' \leqslant Q''$, we can prove that $\dfrac{Q'}{P' + \sqrt{R}} \leqslant \dfrac{1}{2} + \dfrac{\sqrt{R}}{Q} - \dfrac{\sqrt{4R - Q^2}}{2Q}$.

Again, from (ii), if $|Q| > 2\sqrt{R}$,

$Q(Q' - Q'') < 0$

$$i.e., \quad \frac{Q' + Q''}{Q} \gtrless 0 \quad \text{according as } |Q'| \lessgtr |Q''|,$$

$$i.e., \quad \frac{2P' + Q}{2Q} \gtrless 0 \quad \text{according as } |Q'| \lessgtr |Q''|$$

$$i.e., \quad \frac{1}{2} + \frac{\sqrt{R}}{Q} \gtrless \frac{\sqrt{R} - P'}{Q} \text{ according as } |Q'| \lessgtr |Q''|$$

$$i.e., \quad \frac{1}{2} + \frac{\sqrt{R}}{Q} \gtrless \frac{Q'}{P' + \sqrt{R}} \text{ according as } |Q'| \lessgtr |Q''|$$

If $|Q'| = |Q''|$, then $Q' \neq Q''$ and therefore $Q' + Q'' = 0$, which implies $\dfrac{Q'}{P' + \sqrt{R}} = \dfrac{1}{2} + \dfrac{\sqrt{R}}{Q}$.

Thus (v) is proved.

2·2. Having settled the preliminaries, we proceed to define the new continued fraction development as follows :

Let $\dfrac{P + \sqrt{R}}{Q} = \xi_0$ be a surd in the standard form and a the greatest integer less than ξ_0. Then ξ_0 can be represented in one of two forms

$\xi_0 = a + \frac{Q'}{P' + \sqrt{R}}$ (I) or $\xi_0 = a + 1 - \frac{Q''}{P'' + \sqrt{R}}$ (II) where $\frac{P' + \sqrt{R}}{Q'}$ and $\frac{P'' + \sqrt{R}}{Q''}$ are also standard surds.

We call (I) the positive and (II) the negative representation of ξ_0. Choose a or $a+1$ as the partial quotient of the new continued fraction development according as (i) $|Q'| < |Q''|$ or $|Q'| > |Q''|$ if $|Q'| \neq |Q''|$, and (ii) $Q < 0$ or $Q > 0$ if $|Q'| = |Q''|$.

After making the appropriate choice of (I) or (II), we write

$\xi_0 = \frac{P + \sqrt{R}}{Q} = b_0 + \frac{\epsilon_1}{\xi_1}$ where $|\epsilon_1| = 1$, b_0 is an integer, and $\xi_1 = \frac{P_1 + \sqrt{R}}{Q_1} > 1$.

We proceed similarly with ξ_1 for determining the next partial quotient, and so on. Thus $\xi_n = b_n + \frac{\epsilon_{n+1}}{\xi_{n+1}}$ and $\xi_0 = b_0 + \frac{\epsilon_1}{b_1} + \frac{\epsilon_2}{b_2} + \cdots$ad. inf. (1)

This development is obviously unique and we call it the Bhaskara continued fraction (B.*c.f.*), or the nearest square continued fraction, for reasons to be noted presently.

The classical relations connecting P_n, Q_n, P_{n+1}, Q_{n+1} are

$P_{n+1} + P_n = b_n Q_n$ (2) $P^2_{n+1} + \epsilon_{n+1} Q_n Q_{n+1} = R.$ (3)

As in the ordinary theory all P's and Q's are integers, and by Theorem I (iii), the Q's successively diminish in numerical value as long as $|Q| > \sqrt{R}$ and so, ultimately $|Q| < \sqrt{R}$. When once this stage is reached, the P's and Q's thereafter become positive and bounded, $P < 2\sqrt{R}$, $Q < \sqrt{R}$ by Theorem I (iv). Since there can only be a finite number of positive integral P's and Q's which are bounded, while the continued fraction itself is an infinite one (ξ_0 being irrational), the periodicity of the complete quotients and thereby of the partial quotients with the corresponding ϵ's is established.

Thus we establish,

THEOREM II. Every B.*c.f.* development of a quadratic surd is a unique periodic *h.r.c.f.*

Note.—(1) If $\xi_0 = b_0 + \frac{\epsilon_1}{b_1} + \frac{\epsilon_2}{b_2} + \cdots$ (*a*.B.*c.f.*), then $-\xi_0 = -b_0 - \frac{\epsilon_1}{b_1} + \frac{\epsilon_2}{b_2} + \cdots$ is a B.*c.f.* This follows immediately from the manner of the development which takes into account the relative absolute magnitudes and not the signs of the Q's.

(2) From Theorem I (v), it is easily seen that

(i) $\epsilon_{n+1} = 1$, if the fractional part of ξ_n is less than $\frac{1}{2}$ and $Q_n > 0$; and

(ii) $\epsilon_{n+1} = -1$, if the fractional part of ξ_n is greater than $\frac{1}{2}$ and $Q_n < 0$.

(3) From Theorem I (iii), it follows that if $2^{n-1}\sqrt{R} < |Q| < 2^n\sqrt{R}$, then $0 < |Q_m| < \sqrt{R}$ for some value of m ($\geqslant n$) less than $1 + \log_2 |Q| - \frac{1}{2}\log_2 R$.

2·3. *Implications in the Conditions of the Definition of the* B.*c.f.*

If $\xi_0 = \dfrac{P + \sqrt{R}}{Q} = a + \dfrac{Q'}{P' + \sqrt{R}} = a + 1 - \dfrac{Q''}{P'' + \sqrt{R}}$ as in §2·2, we have $|Q'| \lesseqgtr |Q''|$ according as $|QQ'| \lesseqgtr |QQ''|$, *i.e.*, $|P'^2 - R| \lesseqgtr |P''^2 - R|$. (4)

Hence, if we are choosing the lesser of Q′, Q″, we are choosing, in effect, the nearest of the two squares P'^2, P''^2 to R as the basis of our development; and if the two squares are equidistant from R, we can obviously select either; but to avoid ambiguity, we put in the convention that we choose Q′ or Q″ according as Q is less or greater than 0.

Thus, the name ‘nearest square continued fraction’ is justified.

With the help of Theorem I (v), we may give the following alternative *definition*: A number of the form $\dfrac{P + \sqrt{R}}{Q}$ is said to be developed as a ‘nearest square continued fraction’ or a B.*c.f.*, when we assign to each complete quotient, say $\dfrac{P_n + \sqrt{R}}{Q_n}$, a positive or negative representation according as its fractional part is less or greater than $\frac{1}{2} + \dfrac{\sqrt{R}}{Q_n} - \dfrac{\sqrt{4R - Q_n^2}}{2Q_n}$ $\left(\text{or } \frac{1}{2} + \dfrac{\sqrt{R}}{Q_n}\right)$ $|Q_n|$ being less (or greater) than $2\sqrt{R}$.

When the fractional part is equal to $\frac{1}{2} + \dfrac{\sqrt{R}}{Q_n} - \dfrac{\sqrt{4R - Q_n^2}}{2Q_n}$ $\left(\text{or } \frac{1}{2} + \dfrac{\sqrt{R}}{Q_n}\right)$, which we may call critical fractions, the representation to be chosen is positive or negative according as Q_n is negative or positive.

A representation according to the above definition is called a Bhaskara representation (B.R.).

If $\dfrac{P + \sqrt{R}}{Q} = b_0 + \dfrac{\epsilon_1 Q_1}{P_1 + \sqrt{R}}$ be a B.R., where $0 < |Q| < \sqrt{R}$ it implies, by (4) above, $|P_1^2 - R| \leqslant |(P_1 + \epsilon_1 Q)^2 - R|$ (A)

From Theorem I (i), we get $Q_1 - \frac{1}{2}\epsilon_1 Q \leqslant P_1$; if the *l.h.s.*, of this be negative, $|Q_1 - \frac{1}{2}\epsilon_1 Q| \leqslant P_1$, since $Q_1^2 + \frac{1}{4}Q^2 < \frac{1}{2}Q^2 < R$, Q_1 being less than $\frac{1}{2}|Q|$.

Thus, $Q_1 - \frac{1}{2}\epsilon_1 Q \leqslant P_1$ implies $|Q_1 - \frac{1}{2}\epsilon_1 Q| \leqslant P_1$; and *vice versa*. (B)

Squaring both sides of the above inequality, we get

$$Q^2_1 + \tfrac{1}{4} Q^2 \leqslant P^2_1 + \epsilon_1 QQ_1 = R, \text{ i.e., } Q^2_1 + \tfrac{1}{4} Q^2 \leqslant R. \qquad \text{(C)}$$

Conversely, it is easy to see that (C) implies (B).

Hence, (A), (B), (C), are all equivalent to one another, when $0 < |Q| < \sqrt{R}$.

‡ Similarly, we can write down another set of equivalent conditions :

$$|P_1^2 - R| \leqslant |(P_1 + \epsilon'_1 Q_1)^2 - R|. \quad (A'); \quad |Q - \tfrac{1}{2}\epsilon_1 Q_1| \leqslant P_1. \quad (B');$$

$$Q^2 + \tfrac{1}{4} Q^2_1 \leqslant R. \quad (C').$$

It is not difficult to verify that (C) and (C′) imply that P_1 and $|P_1 + \epsilon_1' Q_1|$ (or, $|P_1 + \epsilon_1 Q|$) are such that one is less and the other greater than $\sqrt{R}$. (D)

Further, if one of the equivalent pairs (A), (A′); (B), (B′); (C), (C′) implying (D) holds, the following inequalities are true :

$$P_1 \geqslant \tfrac{1}{2} |Q|, \tfrac{1}{2} Q_1. \quad (E); \quad |P_1 - \sqrt{R}| < |Q|, Q_1. \quad (F)$$

For, (E) is evident when $\epsilon_1 Q = -|Q|$; and when $\epsilon_1 Q = |Q|$ and $|Q| \geqslant Q_1$, (B′) shows $P_1 \geqslant |Q| - \frac{1}{2} Q_1 \geqslant \frac{1}{2}|Q| \geqslant \frac{1}{2} Q_1$; when $\epsilon_1 Q = |Q|$ and $|Q| \leqslant Q_1$, we get the same result from (B). (F) follows immediately from (D), for example, if $P_1 < \sqrt{R}$, then $|P_1 + \epsilon_1 Q|$ and $|P_1 + \epsilon'_1 Q_1|$ are both greater than $\sqrt{R}$ and ϵ'_1 in this case must be $+1$.

That the condition (C′) can co-exist with (C) is clear from the consideration that the Q's in the B.*c.f.* development ultimately become positive and satisfy the conditions (A), (B), or (C). Since the Q's cannot go on perpetually decreasing after they become positive, a stage must come when a Q is not less than its predecessor. Thus, if $Q_1 \geqslant Q$, we get

$$Q^2 + \tfrac{1}{4} Q^2_1 \leqslant Q^2_1 + \tfrac{1}{4} Q^2 \leqslant R.$$

3. Characteristics of the Ultimate Partial and Complete Quotients.

Definition.—A surd in the standard form $\dfrac{P_v + \sqrt{R}}{Q_v}$ is said to be a 'special' surd, when its successor in the B.*c.f* development $\dfrac{P_{v+1} + \sqrt{R}}{Q_{v+1}}$ is such that $Q^2_{v+1} + \frac{1}{4} Q_v^2 \leqslant R, \quad Q_v^2 + \frac{1}{4} Q^2_{v+1} \leqslant R.$

A surd is said to be 'semi-reduced' if it is the successor of a 'special' surd. The successor of a 'semi-reduced' surd is called a 'reduced' surd.

THEOREM III. The conjugate of a semi-reduced surd has its absolute value less than 1.

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‡ $\epsilon'_1 = +1$ or -1 according as $P_1 < \sqrt{R}$ or $P_1 > \sqrt{R}$.

Proof. In the notation of the above definition, the conjugate of the semi-reduced surd $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$ is $\frac{P_{v+1}-\sqrt{R}}{Q_{v+1}}$, whose absolute value is less than 1, by §2·3 (F).

THEOREM IV. A semi-reduced surd is also a special surd.

Proof. Let $\frac{P_v+\sqrt{R}}{Q_v}$ be a special surd, and its B.R. be given by

$$\frac{P_v+\sqrt{R}}{Q_v}=b_v+\frac{\epsilon_{v+1}Q_{v+1}}{P_{v+1}+\sqrt{R}};$$

further, let $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}+\sqrt{R}}$ (B.R.),

It is required to prove that $Q^2_{v+1}+\frac{1}{4}Q_{v+2}{}^2\leqslant R$. (1)

Now, (1) is true when $Q_{v+1}\leqslant Q_{v+2}$, or $Q_{v+2}\leqslant|Q_v|$.

We have, therefore, to consider only the remaining case

$$Q_{v+1}>Q_{v+2}>|Q_v| \qquad (2)$$

Since $P^2_{v+2}-P^2_{v+1}=Q_{v+1}(\epsilon_{v+1}Q_v-\epsilon_{v+2}Q_{v+2})$, and

$P_{v+2}+P_{v+1}=b_{v+1}Q_{v+1}$, we have

$$P_{v+2}-P_{v+1}=(\epsilon_{v+1}Q_v-\epsilon_{v+2}Q_{v+2})/b_{v+1} \qquad (3)$$

If $b_{v+1}=1$, $P_{v+1}=\frac{1}{2}Q_{v+1}-\frac{1}{2}\epsilon_{v+1}Q_v+\frac{1}{2}\epsilon_{v+2}Q_{v+2}<Q_{v+1}-\frac{1}{2}\epsilon_{v+1}Q_v$.

By hypothesis, $P_{v+1}\geqslant Q_{v+1}-\frac{1}{2}\epsilon_{v+1}Q_v$ (4)

which is equivalent to $Q^2_{v+1}+\frac{1}{4}Q_v{}^2\leqslant R$

Thus, there is a contradiction.

Hence $b_{v+1}\geqslant 2$. (4′)

From (3) and (4′),

$$|P_{v+2}-P_{v+1}|\leqslant\tfrac{1}{2}|\epsilon_{v+1}Q_v-\epsilon_{v+2}Q_{v+2}|$$

If $P_{v+2}\leqslant P_{v+1}$, then

$$\begin{aligned}P_{v+2}&\geqslant P_{v+1}+\tfrac{1}{2}\epsilon_{v+1}Q_v-\tfrac{1}{2}\epsilon_{v+2}Q_{v+2}\\&\geqslant Q_{v+1}-\tfrac{1}{2}\epsilon_{v+2}Q_{v+2}\quad\text{by (4)}\end{aligned}$$

which is what we have to prove, being equivalent to (1).

Next, if $P_{v+2}>P_{v+1}$, then

$$P_{v+2}-P_{v+1}\leqslant\tfrac{1}{2}(\epsilon_{v+1}Q_v-\epsilon_{v+2}Q_{v+2}) \qquad (5)$$

i.e., $P_{v+2}+\frac{1}{2}\epsilon_{v+2}Q_{v+2}\leqslant P_{v+1}+\frac{1}{2}\epsilon_{v+1}Q_v$ (5′)

Since both sides of (5) are positive, $\epsilon_{v+2} = -1$ lest (2) should be contradicted.

Further $Q_{v+1} \leqslant P_{v+1} + \frac{1}{2}\epsilon_{v+1}Q_v$, by (4) (6)

Now, two cases may occur ; either

$P_{v+2} + \frac{1}{2}\, \epsilon_{v+2}\, Q_{v+2} \geqslant Q_{v+1}$ or $< Q_{v+1}$

the latter of which will be proved to be impossible.

For, in the latter case,

$P_{v+2} < \frac{1}{2}\, Q_{v+2} + Q_{v+1}$ since $\epsilon_{v+2} = -1$

$< \frac{3}{2}\, Q_{v+1}$ by (2)

$\therefore\ P_{v+1} + P_{v+2} < 2P_{v+2} < 3Q_{v+1}$

so that $b_{v+1} = 1$ or 2.

But, by (4′), $b_{v+1} \geqslant 2$

Hence, $b_{v+1} = 2$.

$\therefore$ From (3), $P_{v+1} = Q_{v+1} - \frac{1}{4}\, \epsilon_{v+1}\, Q_v - \frac{1}{4}\, Q_{v+2}$

$\geqslant Q_{v+1} - \frac{1}{2}\, \epsilon_{v+1}\, Q_v$ by (4)

$\therefore Q_{v+2} \leqslant \epsilon_{v+1}\, Q_v$ which will be impossible if the right-hand side is negative and will contradict (2), if the right-hand side is positive.

Thus, $P_{v+2} + \frac{1}{2}\, \epsilon_{v+2}\, Q_{v+2} < Q_{v+1}$

cannot hold in any case.

This establishes our theorem.

Cor. 1. The successor of a reduced surd is a reduced surd.

Cor. 2. All the complete quotients of a B.*c.f.* are ultimately reduced surds.

Cor. 3. The conjugate of a reduced surd has its absolute value less than 1.

Cor. 4. The partial quotients corresponding to a semi-reduced and therefore a reduced surd are always greater than 1.

For, by (E) § 2·3. $P_{v+2} > \frac{1}{2}\, Q_{v+1}$, and

$P_{v+1} > \frac{1}{2}\, Q_{v+1}$, so that

$P_{v+2} + P_{v+1} > Q_{v+1}$

i.e., $b_{v+1}\, Q_{v+1} > Q_{v+1}$

$\therefore\quad b_{v+1} > 1.$

THEOREM V. A semi-reduced surd is greater than $\frac{\sqrt{5}+1}{2}$.

In the notation of Theorem III, we have to prove

that $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > \frac{\sqrt{5}+1}{2}$.

If $Q_{v+1} < \frac{\sqrt{4R}}{\sqrt{5}}$, $\frac{\sqrt{R}}{Q_{v+1}} > \frac{\sqrt{5}}{2}$ (i)

But, $P_{v+1} > \frac{1}{2} Q_{v+1}$

$$\therefore \quad \frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > \frac{\sqrt{5}+1}{2}.$$

If $P_{v+1} \geqslant Q_{v+1}$, obviously $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > 2 > \frac{\sqrt{5}+1}{2}$. (ii)

If $P_{v+1} < Q_{v+1}$ and $Q_{v+1} > \frac{\sqrt{4R}}{\sqrt{5}}$, (iii)

$$2.118 \cdots = 1 + \frac{\sqrt{5}}{2} > \frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > 1.$$

But $b_{v+1} \geqslant 2$.

Hence, either $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > 2$, or $1 < \frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} < 2$;

in the latter case its representation must be negative, so that its fractional part by Def. § 2·3 is greater than or equal to the critical fraction

$\frac{1}{2} + \frac{\sqrt{R}}{Q_{v+1}} - \frac{\sqrt{4R - Q^2_{v+1}}}{2Q_{v+1}}$, which is greater than $\frac{\sqrt{5}-1}{2}$, when $\frac{\sqrt{R}}{Q_{v+1}} < \frac{\sqrt{5}}{2}$.

In this case,

$$\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} = 2 - \frac{Q_{v+2}}{P_{v+2}+\sqrt{R}}$$
$$= 1 + \left(\text{a fraction greater than } \frac{\sqrt{5}-1}{2}\right)$$
$$> \frac{\sqrt{5}+1}{2}.$$

Thus, in all cases,

$$\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}} > \frac{\sqrt{5}+1}{2}.$$

Cor. 1. A reduced surd is always greater than $\frac{\sqrt{5}+1}{2}$.

Cor. 2. All the complete quotients of a B.*c.f.* are ultimately and therefore in the recurring cycle greater than $\frac{\sqrt{5}+1}{2}$.

Hence, we have

THEOREM VI. The cyclic part of the Bhaskara continued fraction is *canonical.*||

THEOREM VII. If $\frac{P_v + \sqrt{R}}{Q_v}$, $\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}}$ be two successive reduced surds, $|P_{v+1} - P_v| \leqslant Q_v$.

If P_{v+1} and P_v are both greater than $\sqrt{R}$ or both less than $\sqrt{R}$,

$$\begin{aligned} |P_{v+1} - P_v| &= (P_{v+1} \sim \sqrt{R}) \sim (P_v \sim \sqrt{R}) \\ &< \text{the greater of } (P_{v+1} \sim \sqrt{R}),\ (P_v \sim \sqrt{R}) \\ &< Q_v. \end{aligned}$$

If $P_{v+1} > \sqrt{R}$, and $P_v < \sqrt{R}$, again, we have by (D) § 2·3

$$R - (P_{v+1} - Q_v)^2 \nless P^2_{v+1} - R, \text{ and}$$

$$(P_v + Q_v)^2 - R \nless R - P_v^2.$$

Adding, $(P_v + P_{v+1})(P_v - P_{v+1} + 2Q_v) \nless (P_{v+1} + P_v)(P_{v+1} - P_v)$,

so that $2(P_{v+1} - P_v) \leqslant 2Q_v$

i.e., $P_{v+1} - P_v \leqslant Q_v$.

The equality will occur only when

$(P_{v+1} - Q_v)^2 + P_{v+1}^2 = 2R = P_v^2 + (P_v + Q_v)^2$, *i.e.*, when $P^2_{v+1} + P_v^2 = 2R$.

If $P_{v+1} < \sqrt{R}$, $P_v > \sqrt{R}$, we get in the same way $P_v - P_{v+1} \leqslant Q_v$.

In the case of equality, we write

$P_{v+1} + P_v = b_v Q_v$, $P_{v+1} - P_v = \pm Q_v$; so that

$P_{v+1} = (b_v \pm 1)\, Q_v/2$, $P_v = (b_v \mp 1)\, Q_v/2$,

$4R = (P_{v+1} + P_v)^2 + (P_{v+1} - P_v)^2 = (b_v^2 + 1)\, Q_v^2$ which implies that Q_v is even; and $\left| \frac{R - P_v^2}{Q_v} \right| = \frac{1}{2} b_v Q_v$.

Further, the surds being in the standard form, $\frac{R - P^2_v}{Q_v}$, P_v, Q_v, must have their highest common factor $\frac{1}{2} Q_v$ equal to unity.

Hence, $Q_v = 2$, $R = b_v^2 + 1$, $P_v = b_v \mp 1$, $P_{v+1} = b_v \pm 1$, $Q_{v+1} = b_v$; where $\frac{P_v + \sqrt{R}}{Q_v} = b_v + \frac{\epsilon_{v+1} Q_{v+1}}{P_{v+1} + \sqrt{R}}$.

Thus, $\frac{b_v - 1 + \sqrt{b_v^2 + 1}}{2} = b_v - \frac{b_v}{b_v + 1 + \sqrt{b_v^2 + 1}}$; or

|| *Vide* p. 170, *Die Lehre von den Kettenbrüchen*, by O. Perron, 1929.

$$\frac{b_v + 1 + \sqrt{b_v^2+1}}{2} = b_v + \frac{b_v}{b_v - 1 + \sqrt{b_v^2 + 1}}$$ of which the first alone is a B.R.

Therefore $|P_{v+1} - P_v| < Q_v$ in all cases except when $P_v = b_v - 1$, $b_v > 2$, $R = b_v^2 + 1$, $Q_v = 2$.

For, if $b_v = 2$, $\frac{P_v + \sqrt{R}}{Q_v} \left(= \frac{1 + \sqrt{5}}{2} \right)$ is not even semi-reduced.

N.B.—The above proof applies even when the first of the given surds is semi-reduced.

We reserve other properties of the continued fraction for a separate communication.

THEORY OF THE NEAREST SQUARE CONTINUED FRACTION*

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4. *Special Critical Fractions*

4·1. IN §2 of our previous communication† we have called the surds (i) $\frac{1}{2}+\frac{\sqrt{R}}{Q}-\frac{\sqrt{4R-Q^2}}{2Q}$ $(|Q|<2\sqrt{R})$, and (ii) $\frac{1}{2}+\frac{\sqrt{R}}{Q}$ $(|Q|>2\sqrt{R})$, critical fractions, since they decide the nature of the representations to be assigned to $\frac{P+\sqrt{R}}{Q}$ in a B.c.f. development. Ambiguities arise when (iii) $\frac{P+\sqrt{R}}{Q}-\frac{1}{2}-\frac{\sqrt{R}}{Q}+\frac{\sqrt{4R-Q^2}}{2Q}=\frac{P}{Q}-\frac{1}{2}+\frac{\sqrt{4R-Q^2}}{2Q}=$ an integer, $(|Q|<2\sqrt{R})$ which implies $4R-Q^2=t^2$, $\frac{2P+t}{Q}$ an odd integer, Q, t both even integers, and R is the sum of two squares; and (iv) $\frac{P+\sqrt{R}}{Q}-\frac{1}{2}-\frac{\sqrt{R}}{Q}=\frac{P}{Q}-\frac{1}{2}=$ an integer $(|Q|>2\sqrt{R})$; but these cases have been circumvented by appropriate conventions.

If $\frac{P+\sqrt{R}}{Q}$ be a special surd with $\frac{P_1+\sqrt{R}}{Q_1}$ as its successor, and $R=Q_1^2+\frac{1}{4}Q^2>Q^2+\frac{1}{4}Q_1^2$, then it is easily seen that the fractional part of $\frac{P+\sqrt{R}}{Q}$ in its positive representation is equal to the corresponding critical fraction which takes the special form $\frac{1}{2}+\frac{\sqrt{R}-Q_1}{Q}$, where $Q_1>|Q|$.

Definition.—A proper fraction of the form $\frac{q-p+\sqrt{R}}{2q}$ (R a non-square positive integer) is called a special critical fraction when $R=p^2+q^2$, and $p>2q>0$.

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* This is a continuation of the memoir published in the *Journal of the Mysore University*, Vol. I, Part II, pp. 21–32.

† See *ibid.*, Vol. I, Part II, p. 26.

4·2. THEOREM VIII. If $\frac{P_{v-1}+\sqrt{R}}{Q_{v-1}}$ is a special surd with successors $\frac{P_v+\sqrt{R}}{Q_v}$, $\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}$, $\frac{P_{v+2}+\sqrt{R}}{Q_{v+2}}$ in a B. c.f. development, then $\frac{P_{v+1}+\sqrt{R}}{Q_v}$ is a successor of $\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}$ in all cases except when $R=Q_v^2+\frac{1}{4}Q^2_{v-1}$.

Let
$$\frac{P_{v+1}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}+\sqrt{R}}.$$

Then,
$$\frac{P_{v+1}-\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+2}Q_{v+2}}{P_{v+2}-\sqrt{R}}$$

i.e.,
$$\frac{\epsilon_{v+1}Q_v}{P_{v+1}+\sqrt{R}}=b_{v+1}-\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}$$

Hence
$$\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}=b_{v+1}+\frac{\epsilon_{v+1}Q_v}{P_{v+1}+\sqrt{R}} \tag{1}$$

where
$$Q_v^2+\tfrac{1}{4}Q^2_{v+1}<R,\quad Q^2_{v+1}+\tfrac{1}{4}Q_v^2\leqslant R.$$

Therefore $\frac{P_{v+1}+\sqrt{R}}{Q_v}$ will be a Bhaskara successor of $\frac{P_{v+2}+\sqrt{R}}{Q_{v+1}}$ in all cases except when

$$Q_v^2+\tfrac{1}{4}Q^2_{v+1}=R \text{ and } \epsilon_{v+1}=1,$$

which will violate our convention in the ambiguous case.

Similarly,
$$\frac{P_{v+1}+\sqrt{R}}{Q_v}=b_v+\frac{\epsilon_v Q_{v-1}}{P_v+\sqrt{R}} \tag{2}$$

where
$$Q^2_{v-1}+\tfrac{1}{4}Q_v^2\leqslant R,\quad Q_v^2+\tfrac{1}{4}Q^2_{v-1}\leqslant R.$$

If $\epsilon_{v+1}=1$, and $Q_v^2+\frac{1}{4}Q^2_{v+1}=R$, we have

$$P^2_{v+1}=R-Q_vQ_{v+1}=(Q_v-\tfrac{1}{2}Q_{v+1})^2;$$

since $Q_v^2+\frac{1}{4}Q^2_{v+1}=R>Q^2_{v+1}+\frac{1}{4}Q_v^2$,

$$Q_v>Q_{v+1},\text{ and } P_{v+1}=Q_v-\tfrac{1}{2}Q_{v+1} \tag{3}$$

Hence,
$$\frac{P_{v+1}+\sqrt{R}}{Q_v}=\frac{\sqrt{R}+Q_v-\frac{1}{2}Q_{v+1}}{Q_v}$$
$$=2+\frac{\sqrt{R}-Q_v-\frac{1}{2}Q_{v+1}}{Q_v}$$
$$=2-\frac{Q_{v+1}}{\sqrt{R}+Q_v+\frac{1}{2}Q_{v+1}} \tag{4}$$

Comparing (2) and (4), we have $\epsilon_v=-1$, $Q_{v+1}=Q_{v-1}$, $P_v-Q_v=\frac{1}{2}Q_{v+1}$, and therefore $\frac{1}{4}Q^2_{v-1}+Q_v^2=R$.

Conversely, if $Q_v^2 + \frac{1}{4} Q^2_{v-1} = R$, we see that $Q_v > |Q_{v-1}|$, $P_v = Q_v - \frac{1}{2} \epsilon_v Q_{v-1}$, where $\epsilon_v Q_{v-1}$ is negative, and

$$\frac{P_v + \sqrt{R}}{Q_v} = 2 + \frac{\sqrt{R} + \frac{1}{2}|Q_{v-1}| - Q_v}{Q_v} = 2 + \frac{|Q_{v-1}|}{\sqrt{R} - \frac{1}{2}|Q_{v-1}| + Q_v},$$

so that $\epsilon_{v+1} = 1$, $Q_{v+1} = |Q_{v-1}|$, and $Q_v^2 + \frac{1}{4} Q^2_{v+1} = R$.

Hence, $\frac{P_{v+1} + \sqrt{R}}{Q_v}$ will fail to be a successor of $\frac{P_{v+2} + \sqrt{R}}{Q_{v+1}}$, when and only when $Q_v^2 + \frac{1}{4} Q^2_{v-1} = R$.

4·3. THEOREM IX: *Two different semi-reduced surds cannot have the same Bhaskara successor unless they are conjugates of* $-g$ *and* $1-g$, g *being any special critical fraction.*

If possible, let two different semi-reduced surds

$\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, $\xi_v' = \frac{P_v' + \sqrt{R}}{Q'_v}$ have the same successor ξ_{v+1} (where $P_v' > P_v$), while the predecessor of ξ_v is ξ_{v-1}.

$$\text{Then } \frac{P_v + \sqrt{R}}{Q_v} \pm \frac{P_v' + \sqrt{R}}{Q'_v} = \text{an integer.} \qquad (1)$$

Hence $Q_v = \pm Q'_v$ the irrational part being equated to zero.

But Q_v, Q_v' are both positive, the surds being semi-reduced; hence $Q_v = Q'_v$ and the sign to be chosen in (1) is negative, so that

$$P_v - P'_v \equiv 0 \pmod{Q_v} \qquad (2)$$

Arguing as in the proof of Theorem VII* replacing therein P_{v+1} by P'_v but omitting the consideration $P'_v < R^{\frac{1}{2}}$, $P_v > R^{\frac{1}{2}}$, which has obviously no application in the present context, we get $P'_v - P_v \leqslant Q_v$. (3)

From (2) and (3), $P'_v = P_v$, or $P'_v - P_v = Q_v$, and in the latter case, $P'^2_v + P_v^2 = 2R$, from which we derive $P_v = |Q_{v-1}| - \frac{1}{2} Q_v$, $P'_v = |Q_{v-1}| + \frac{1}{2} Q_v$, $R = Q^2_{v-1} + \frac{1}{4} Q_v^2$, since we may put $P_v^2 = R - |Q_{v-1}| Q_v$.

Thus the two surds which have the same successor are of the form

$$\xi_v = \frac{|Q_{v-1}| - \frac{1}{2} Q_v + \sqrt{R}}{Q_v}; \quad \xi_v' = \frac{|Q_{v-1}| + \frac{1}{2} Q_v + \sqrt{R}}{Q_v} = 1 + \xi_v;$$

where $R = Q^2_{v-1} + \frac{1}{4} Q_v^2$, Q_v is even and less than $|Q_{v-1}|$.

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* See *Journal of the Mysore University*, Vol I, Part II, page 31,

Obviously, $\dfrac{\frac{1}{2}Q_v - |Q_{v-1}| + \sqrt{R}}{Q_v}$ is a special critical fraction, g say. ξ_v is the conjugate of $-g$ and ξ_v' is the conjugate of $(1-g)$. This proves the proposition.

4·4. THEOREM X: *If* g *be a special critical fraction, then* g^{-1} *has no Bhaskara predecessor,* $(1-g)^{-1}$ *is semi-reduced, and the Bhaskara successors of* g^{-1} *and* $(1-g)^{-1}$ *are respectively the conjugates of* $1-g$ *and* $-g$; *the conjugate of* $1-g$ *has no semi-reduced predecessor, while the conjugate of* $-g$ *has a unique semi-reduced predecessor.*

Let $g = \dfrac{q-p+\sqrt{p^2+q^2}}{2q}$, $(p > 2q > 0)$. Then a predecessor of g^{-1} or $(1-g)^{-1}$ will be of the form $a \pm g$, where a is an integer.

Put $\dfrac{P+\sqrt{R}}{Q} = a+g = a + \dfrac{p}{p-q+\sqrt{R}} = a+1-(1-g) = a+1-\dfrac{p}{p+q+\sqrt{R}}$,

where $R = p^2+q^2$.

Then $Q = 2q < p < R^{\frac{1}{2}}$; $p^2 + \frac{1}{4}Q^2 = R > Q^2 + \frac{1}{4}p^2$, so that $\dfrac{P+\sqrt{R}}{Q}$ is a special surd.

Hence g^{-1} has no predecessor of the form $a+g$, while $(1-g)^{-1}$, has one of the form $a+1-(1-g)$.

Similarly, it can be shown that g^{-1} has no predecessor of the form $a-g$, while $(1-g)^{-1}$ has a predecessor of the form $a-1+(1-g)$.

Now $$g^{-1} = \frac{p-q+\sqrt{R}}{p} = 1 + \frac{p}{q+\sqrt{R}} = 2 - \frac{2q}{p+q+\sqrt{R}}$$

$$= 2 - \frac{1}{\text{conjugate of }(1-g)};$$

$$\text{and } (1-g)^{-1} = \frac{p+q+\sqrt{R}}{p} = 3 - \frac{3p-4q}{2p-q+\sqrt{R}}$$

$$= 2 + \frac{2q}{p-q+\sqrt{R}} = 2 + \frac{1}{\text{conjugate of }(-g)}.$$

Since $2q < p < 3p-4q$, the Bhaskara successors of g^{-1} and $(1-g)^{-1}$ are respectively the conjugates of $(1-g)$ and $-g$.

Any predecessor of the conjugate of $(1-g)$ must be of the form $a \pm \dfrac{p+q-\sqrt{R}}{p}$, where a is an integer. For a semi-reduced predecessor, $a + \dfrac{p+q-\sqrt{R}}{p}$ is inadmissible and a must be an integer such that $p(a-1)-q > 0$, and $(pa-p-q)^2$ is nearest to R; all these conditions are satisfied only when $a = 2$, for it can be easily verified that $p - q < \sqrt{R}$, $pa - p - q > \sqrt{R}$ when $a > 2$, and $R - (p - q)^2 < (2p - q)^2 - R$, when $p > 2q$. Thus the only possible semi-reduced predecessor of the conjugate of $1 - g$ is g^{-1}. But since g^{-1} has no Bhaskara predecessor, it cannot be semi-reduced.

Similarly, the possible semi-reduced predecessors of the conjugate of $-g$ must be of the form $\dfrac{pa - p + q + \sqrt{R}}{p}$, where a is an integer such that $pa - p - q > 0$, and $(pa - p - q)^2$ is nearest to R. Obviously $a = 2$, since when $a \geqslant 2$, $pa - p + q > \sqrt{R}$, and when $a = 1$, $q < \sqrt{R}$, while $(p + q)^2 - R < R - q^2$. Thus the possible semi-reduced predecessor is $(1 - g)^{-1}$, which is certainly semi-reduced with a 'special' surd as its predecessor.

Hence the proposition is proved.

Cor. 1. Two different reduced surds cannot have the same successor.

Cor. 2. Neither the conjugate of $-g$ nor that of $(1 - g)$ can be the successor of a standard surd of the form $\dfrac{\sqrt{R}}{Q}$.

5. *Pure Recurring Bhaskara Continued Fractions*

5·1 *Definition.*—A pure recurring B.c.f. is one in which the complete quotients recur from the first.

We have already seen that the complete quotients in a B.c.f. development are ultimately reduced surds. Hence a pure recurring B.c.f. is equal to a reduced surd.

The converse of this will now be proved.

5·2. THEOREM XI: *The Bhaskara development of a reduced surd is a pure recurring half-regular continued fraction.*

Let $\xi_0 = \dfrac{P_0 + \sqrt{R}}{Q_0}$ be a reduced surd and if possible, let its B.c.f. development be the periodic h.r.c.f.

$$b_0 + \frac{\epsilon_1}{b_1 +} \cdots + \frac{\epsilon_{k-1}}{b_{k-1} +} \frac{\epsilon_k}{\underset{*}{b_k} +} \cdots\cdots + \frac{\epsilon_{k+n-1}}{\underset{*}{b_{k+n-1}}},$$

where $\xi_{k+v} = \xi_{k+v+tn}$ $(v = 0, 1 \cdots\cdots, n - 1)$, t a positive integer, and $b_{k+v} = b_{k+v+tn}$.

Since ξ_0 is reduced, ξ_{k-1} and ξ_{k+n-1} are also reduced; but their respective successors ξ_k and ξ_{k+n} are equal.

By Theorem X, Cor. (1), therefore, $\xi_{k-1} = \xi_{k+n-1}$.

If $\epsilon_{k-1} \neq \epsilon_{k+n-1}$, then $\xi_{k-2} \neq \xi_{k+n-2}$ which will contradict Theorem X, Cor. (1), so that $\epsilon_{k-1} = \epsilon_{k+n-1}$, *i.e.*, the recurrence begins one step earlier. This process can be evidently continued backwards until ξ_0 is reached. The first complete quotient therefore recurs and the h.r.c.f. is a pure recurring one, of the form $\dot{b}_0 + \frac{\epsilon_1}{b_1 +} \cdots \cdots + \frac{\epsilon_k}{\dot{b}_0}$.

5·3. THEOREM XII: *The B.c.f. development of the standard surd* $\frac{\sqrt{R}}{Q}$ (>1) *has only one term in the acyclic part.*

Proof: Let $\xi_0 = \frac{\sqrt{R}}{Q} = b_0 + \frac{\epsilon_1}{\xi_1}$ (a B.R.), where $\xi_1 = \frac{P_1 + \sqrt{R}}{Q_1}$.

Then $$P_1 = b_0 Q,\ \epsilon_1 QQ_1 = R - P_1^2;$$

$\frac{\sqrt{R}}{Q}$ being in the standard form, we may write $R = QQ'$, where Q, Q' are positive integers having no common factor; hence $\epsilon_1 Q_1 = Q' - b_0^2 Q$.

By Theorem I, since $Q < \sqrt{R}$, P_1, Q_1 are positive and $|Q_1 - \frac{1}{2}\epsilon_1 Q| < P_1$. (1)

When $Q < \frac{1}{2} Q_1$, and $\epsilon_1 = 1$, $\frac{1}{2} Q_1 - Q < Q_1 - \frac{1}{2} Q < P_1$ by (1).

We shall now prove that $|Q - \frac{1}{2}\epsilon_1 Q_1| < P_1$, which is equivalent to

$$Q - \tfrac{1}{2} Q' + \tfrac{1}{2} b_0^2 Q < b_0 Q,$$

i.e., $$Q(1 - b_0 + \tfrac{1}{2} b_0^2) < \tfrac{1}{2} Q',$$

i.e., $$(b_0 - 1)^2 < \frac{Q'}{Q} - 1, \text{ when } Q > \tfrac{1}{2} Q_1 \qquad (2)$$

If $\epsilon_1 = 1$, $b_0 Q < \sqrt{R} = \sqrt{QQ'}$, *i.e.*, $b_0^2 < \frac{Q'}{Q}$, so that

$$(b_0 - 1)^2 < b_0^2 - 1 < \frac{Q'}{Q} - 1. \qquad (3)$$

If $\epsilon_1 = -1$, we have from (1), $Q_1 + \frac{1}{2} Q < P_1$,

i.e., $$b_0^2 Q - Q' + \tfrac{1}{2} Q < b_0 Q,$$

i.e., $$b_0^2 - b_0 + \tfrac{1}{2} < Q' / Q.$$

When $\epsilon_1 = -1$, $b_0 > 1$, and so $(b_0 - 1)^2 + 1 < b_0^2 - b_0 + \frac{1}{2}$;

hence $$(b_0 - 1)^2 < \frac{Q'}{Q} - 1. \qquad (4)$$

Thus, in all cases, $$|Q - \tfrac{1}{2}\epsilon_1 Q_1| < P_1 \qquad (5)$$

From (1) and (5), $\frac{\sqrt{R}}{Q}$ is a special surd, and therefore ξ_1 is a semi-reduced surd, ξ_2 is a reduced surd and the period of recurrence must begin at least from ξ_2, the successor of ξ_1.

By Theorem X, Cor. (2), ξ_1 cannot be the conjugate of $-g$ or $1-g$, where g is a special critical fraction. ξ_1 is, therefore, the unique semi-reduced predecessor of ξ_2. Hence ξ_1 must recur.

Further, ξ_0 cannot recur; for if $\xi_0 = \xi_{n+1}$ (say), then $P_{n+1} = 0$, and $Q_n Q_{n+1} = R$, an impossible relation when Q_n, Q_{n+1} are each less than $\sqrt{R}$.

Hence the recurring period begins from ξ_1 and the B.c.f. development of $\frac{\sqrt{R}}{Q}$ has one and only one term in the acyclic part.

Cor.—b_0 is such that $b_0^2 Q^2$ is the nearest to R among the square multiples of Q^2.

5·4. THEOREM XIII: *If* g *be a special critical fraction, then* $(1-g)^{-1}$ *develops as a pure recurring B.c.f.*

We know that $(1-g)^{-1}$ is of the form $\frac{p+q+\sqrt{R}}{p}$, where $p > 2q > 0$, $R = p^2 + q^2$. It is sufficient for our purpose to prove that there exists a Bhaskara predecessor of $(1-g)^{-1}$ which is semi-reduced, and the rest will follow from Theorem XI.

As we have seen already in Theorem X, a semi-reduced predecessor of $(1-g)^{-1}$ must be of the form $\frac{(2n-1)q-p+\sqrt{R}}{2q}$, where n is an integer ($\geqslant 2$) and $(2n-1)q - p > 0$, such that its Bhaskara predecessor is a special surd of the form

$$\mu + \frac{2q\epsilon}{(2n-1)q-p+\sqrt{R}} = \mu + \frac{p-(2n-1)q+\sqrt{R}}{\epsilon\{(2n^2-2n)q-p(2n-1)\}},$$

μ being any integer and $\epsilon = \pm 1$.

The condition for special surds gives

$$|2q - \tfrac{1}{2}(2n-1)p + q(n^2-n)| < (2n-1)q - p, \text{ and} \qquad (1)$$

$$|q - p(2n-1) + q(2n^2 - 2n)| < (2n-1)q - p. \qquad (2)$$

We have to consider four cases:

(i) $2q - \frac{1}{2}(2n-1)p + q(n^2-n) > 0$, $q - p(2n-1) + q(2n^2-2n) > 0$;

(ii) ,, ,, > 0, ,, ,, < 0;

(iii) ,, ,, < 0, ,, ,, $\geqslant 0$;

(iv) ,, ,, < 0, ,, ,, < 0.

In case (i), since $p/q = \frac{n^2-n+2}{n-\frac{1}{2}} = \frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$ is impossible, simultaneous equality has to be excluded.

The upper limits (U) and the lower limits (L) of p/q corresponding to the four cases are as follows:

Case	U		L	
(i)	$\frac{n^2-n+2}{n-\frac{1}{2}}$,	$\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$*	$\frac{n^2-3n+3}{n-\frac{3}{2}}$,	$n-1$*
(ii)	n ,	$\frac{n^2-n+2}{n-\frac{1}{2}}$*	$\frac{n^2-3n+3}{n-\frac{3}{2}}$,	$\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$*
(iii)	$\frac{n^2+n+1}{n+\frac{1}{2}}$,	$\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$*	$n-1$,	$\frac{n^2-n+2}{n-\frac{1}{2}}$*
(iv)	$\frac{n^2+n+1}{n+\frac{1}{2}}$,	n*	$\frac{n^2-n+2}{n-\frac{1}{2}}$*,	$\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}$

the starred expressions signifying lesser upper limits (U) and greater lower limits (L) when $n > 4$.

Case (iii) is impossible since the lesser upper limit is obviously less than the greater lower limit; since $p > 2q$, $n = 2$ is impossible in all the four cases; when $n = 3, 4$, the limits for p/q are respectively (2 and 13/5) and (3, 25/7) in case (i) and (13/5 and 3) and (25/7, 4) in case (ii), while case (iv) is inapplicable.

For integral $n > 4$, the first two and the last case are applicable, in order, for the values of p/q in the intervals I_1, I_2, I_3 corresponding respectively to $\left(n-1, \frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}\right)$, $\left(\frac{n^2-n+\frac{1}{2}}{n-\frac{1}{2}}, \frac{n^2-n+2}{n-\frac{1}{2}}\right)$, $\left(\frac{n^2-n+2}{n-\frac{1}{2}}, n\right)$, closed on the right and open to the left.

Thus, for every value of p/q greater than 2, we can always fix up a unique value of n also greater than 2, since p/q is bound to lie in one and only one of the rational intervals (closed on the right and open on the left), (2, 13/5), (13/5, 3), (3, 25/7), (25/7, 4), (4, 41/9), (41/9, 44/9), (44/9, 5) and so on, which cover the entire set of rational numbers greater than 2. This proves our theorem.

5·5. Before discussing further the properties of the recurring B.c.f., we require certain lemmas on the behaviour of unit partial quotients in simple continued fractions.

Lemma (1): If $\xi = \frac{P+\sqrt{R}}{Q}$ develops as a pure recurring simple continued fraction with a set of successive unit partial quotients preceded and followed by other partial quotients, then the denominators of the complete quotients corresponding to the unit partial quotients other than the first and the last of the set are less than $\sqrt{R}$.

Let $\xi = \frac{P+\sqrt{R}}{Q} = \underset{\times}{a_0} + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_r+}\frac{1}{1+}\frac{1}{1+}\cdots+$

$$+\frac{1}{1+}\frac{1}{a_{r+n+1}+}\cdots\frac{1}{\underset{\times}{a_p}}$$

$$= (\underset{\times}{a_0},\ a_1\ a_2,\ \cdots\ a_r, 1_{[n]},\ a_{r+n+1}\ \cdots\ \underset{\times}{a_p}),\ \text{and}$$

$$\xi_{r+v} = \frac{P_{r+v}+\sqrt{R}}{Q_{r+v}} = (\underset{\times}{1}_{[n-v+1]},\ a_{n+r+1},\ \cdots\ a_p,\ a_0,\ a_1,\ \cdots\ a_r,\ \underset{\times}{1}_{[v-1]})$$

$= f$, (say) (1)

By* Galois's theorem of inverse periods,

$$\frac{Q_{r+v}}{-P_{r+v}+\sqrt{R}} = (\underset{\times}{1}_{[v-1]},\ a_r,\ \cdots\ a_0,\ a_p,\ \cdots\ a_{n+r+1},\ \underset{\times}{1}_{[n-v+1]}).$$

Hence, $$\frac{-P_{v+r}+\sqrt{R}}{Q_{v+r}} = (0,\ \underset{\times}{1}_{[v-1]},\ \cdots\ \underset{\times}{1}_{[n-v+1]}) = f',\ \text{(say)}. \qquad (2)$$

Adding (1) and (2), $\frac{2\sqrt{R}}{Q_{r+v}} = f+f'$, so that

$Q_{r+v} \lesseqgtr \sqrt{R}$, according as $f \gtreqless (2-f') = f''$ (say).

But, $$2-f' = (2-(0,\ \underset{\times}{1}_{[v-1]}\ \cdots\ \underset{\times}{1}_{[n-v+1]}))$$

$$= (1,\ 1,\ 0,\ 1_{[v-2]},\ \cdots\cdots)$$

$$= (1,\ 2,\ 1_{[v-3]},\ \cdots\cdots)$$

If $n-1 > v \geqslant 3$, the second complete quotient of f is less than the corresponding complete quotient of f'' and therefore $f > f''$, implying $Q_{r+v} < \sqrt{R}$. If $v=2$ and $n \geqslant 3$, we have $f'' = 2-(0,\ 1,\ a_r,\ \cdots) = (1,\ 1+a_r,\ \cdots) < f$ and again $Q_{r+v} < \sqrt{R}$.

Thus for all values of v greater than 1 and less than n (>3), $Q_{r+v} < \sqrt{R}$.

The lemma is therefore proved.

Cor. (1).—If $n > 2$, and $a_r > 2$, then $Q'_{r+1} > \sqrt{R}$; if $n > 2$, and $a_{r+n+1} > 2$ then $Q'_{r+n} > \sqrt{R}$.

Cor. (2).—If $n=2$, and $1+a_r < a_{r+n+1}$, then $Q'_{r+1} < \sqrt{R}$ and $Q'_{r+n} > \sqrt{R}$; the inequalities are reversed when $1+a_{r+n+1} < a_r$, while both the Q's are less than $\sqrt{R}$ if $a_r = a_{r+n+1}$.

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* *Vide* pp. 82–85. *Die Lehre von den Kettenbrüchen*, by O. Perron, 1929.

Lemma (2): In the simple continued fraction development of a surd of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, p, q, being integers such that $p > 2q > 0$, there cannot occur a complete quotient of the same form more than once in the recurring period; when such a complete quotient does occur, the recurring period is symmetric, with an even number of terms, which include a central set of an even number of unit partial quotients.

Let $\xi_0 = \frac{P_0+\sqrt{R}}{Q_0} = \frac{q+\sqrt{p^2+q^2}}{p}$, with $\xi_v \left(= \frac{P_v+\sqrt{R}}{Q_v}\right)$ as the v-th successor of ξ_0. Let $\bar{\xi}_0$ be the conjugate of ξ_0. Then, $\xi_0\bar{\xi}_0 = -1$;
$1 < \xi_0 < \frac{1+\sqrt{5}}{2} = (1, 1, \cdots) = (\dot{1})$ or (1_∞) and $-1 < \bar{\xi}_0 < 0$. (1)

By a well-known theorem of Galois, the simple continued fraction for ξ_0 has a pure recurring period $(\dot{a}_0, a_1, \cdots \dot{a}_n)$, say.

From (1), $a_0 = 1$ and if a_m is the first partial quotient greater than 1, m must be odd; for, if m be even, we have successively
$(a_m, \cdots) > (1_\infty)$; $(1, a_m, \cdots) < (1_\infty)$; $(1_{[2]}, a_m, \cdots) > (1_\infty)$; $\cdots$, $(\dot{1}_{[m]}, a_m, \cdots \dot{a}_n) > (1_\infty)$, which contradicts (1).

Hence $$\xi_0 = (\dot{1}_{[m]}, a_m, \cdots \dot{a}_n). \quad (2)$$

Again, $$\xi_0 = -1/\bar{\xi}_0 = (\dot{a}_n, \cdots \dot{1}_{[m]}). \quad (3)$$

Comparing (2) and (3), we have $a_n = a_{n-1} = \cdots = a_{n-m+1} = 1$, $a_m = a_{n-m}$; *i.e.*, the period is a symmetric one, beginning and ending with an odd number of unit partial quotients.

The comparison of the complete quotients in (2) and (3) gives
$$\frac{P_v+\sqrt{R}}{Q_v} = \frac{P_{n+1-v}+\sqrt{R}}{Q_{n-v}}, \ (v \leq n), \quad i.e., \quad P_v = P_{n-v+1},\ Q_v = Q_{n-v}.$$
If $Q_v = Q_{v-1}$, then $$\xi_v = \frac{P_v+\sqrt{R}}{Q_v} = \frac{P_{n+1-v}+\sqrt{R}}{Q_{v-1}}$$
$$= \frac{P_{n+1-v}+\sqrt{R}}{Q_{n+1-v}} = \xi_{n+1-v}$$

and so $v = n+1-v$, *i.e.*, $v = \frac{n+1}{2}$, which implies that n should be odd.

Thus, only when n is odd, $Q_{\frac{n+1}{2}} = Q_{\frac{n-1}{2}}$ and these are the only consecutive Q's which can be equal to each other. (4)

If a complete quotient, say, ξ_v should be of the same form as ξ_0, its simple continued fraction development should have the same properties. Writing $Q_0, Q_1 \cdot \cdot \cdot \cdot Q_n$ round a circle at the verticies of a regular polygon of $n+1$ sides, we find that they arrange themselves symmetrically about a diameter, such that the Q's symmetrically placed about this diameter are also equal, since $Q_v = Q_{n-v}$.

The symmetry of the Q's corresponding to ξ_v imply that $Q_v = Q_{v-1}$ just as $Q_0 = Q_n$. From (4) we see that this can happen only once and so, there cannot be more than one ξ_v of the same form as ξ and it occurs when n is odd and $v = \frac{n+1}{2}$. In this case we realise the same symmetry of Q's starting from $Q_{\frac{n+1}{2}}$, going round the circle and ending with $Q_{\frac{n-1}{2}}$ as in the first set $(Q_0, Q_1, \cdot \cdot \cdot \cdot Q_n)$.

This proves the existence of ξ_v of the same form as ξ_0, only when $Q_{\frac{n}{2} \pm r} = 1$, where $r = 1, 3, 5 \cdot \cdot \cdot (2k-1)$, and k, n are both odd.

Hence, if ξ_0 should have a remote successor of the same form as itself in the recurring period of its simple continued fraction development, then the recurring period must consist of an even number of symmetrically disposed partial quotients including an initial, a central and a final set of unit partial quotients. In order that the recurring cycle may not lose its character as a primitive period, it is necessary that the first half of the cycle is not itself symmetrical.

Example.—$\frac{27+\sqrt{27^2+82^2}}{82} = (\overset{\times}{1}, 2, 1_{[6]}, 2, \overset{\times}{1})$ has a remote successor within the recurring period of the same form $\frac{37+\sqrt{37^2+78^2}}{78}$.

Lemma (3): If the standard surd of the form $\frac{\sqrt{R}}{Q_0}$ have in its simple continued fraction development a complete quotient of the form $\frac{q+\sqrt{R}}{p}$, where $R = p^2 + q^2$, $p > 2q > 0$, then the symmetric portion of the recurring period of partial quotients will include a central even number, of the form $4n-2$, of unit partial quotients; and there cannot occur any other complete quotient of a similar form within the recurring period, which must consist of an odd number of terms.

Conversely, if any simple continued fraction development of the standard surd $\frac{\sqrt{R}}{Q_0}$ has in its recurring period an odd number of partial quotients with a central even number $(4n-2)$ of unit partial quotients, in the symmetric part, then $R = p^2 + q^2$, $p > 2q > 0$ and the complete quotient $\frac{q+\sqrt{R}}{p}$ occurs just once in the recurring period.

Let $$\frac{\sqrt{R}}{Q_0} = (a_0, \underset{\times}{a_1}, a_2, \cdots a_{k-1}, \underset{\times}{2a_0}) \tag{1}$$

From Lemma (2), a complete quotient, say ξ_v, of the form in question in (1) cannot have either a_1 or $2a_0$ (obviously $\neq 1$) as its first partial quotient so that we may write $\xi_v = (\underset{\times}{a_v}, \cdots \underset{\times}{a_{v-1}})$, where $a_v \neq a_1$ or $2a_0$. From the equality of the first and last Q's in ξ_v, we must have $Q_v = Q_{v-1}$ in (1), which implies, by a well-known theorem of Muir,* that k is odd and $v = \frac{k+1}{2}$; and in this case, it is easily seen that $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, and $R = P_v^2 + Q_v^2$.

Further, there cannot be another complete quotient of the same form in the recurring period, since it is possible only when the number of terms in the recurring period is even.

We infer therefore that $\xi_{\frac{k+1}{2}} = (\underset{\times}{a_{\frac{k+1}{2}}}, \cdots a_{k-1}, 2a_0, a_1, \cdots \underset{\times}{a_{\frac{k+1}{2}}})$, where an odd number of unit partial quotients must begin with $a_{\frac{k+1}{2}}$ and also an equal odd number of such partial quotients end with $a_{\frac{k-1}{2}}$.

Thus $\frac{\sqrt{R}}{Q_0}$ must contain in its period an even number, of the form $4n-2$, of unit partial quotients in the centre of the symmetric portion, as, for example, $\sqrt{58} = (7, \underset{\times}{1_{[6]}}, \underset{\times}{14})$; $\sqrt{97} = (9, \underset{\times}{1}, 5, 1_{[6]}, 5, 1, \underset{\times}{18})$.

In this case, $\xi_{\frac{k+1}{2}}$ is of the form $\frac{q+\sqrt{p^2+q^2}}{p} < (1_\infty)$, as the continued fraction begins with an odd number of unit partial quotients.

Hence, $\frac{q+\sqrt{p^2+q^2}}{p} < \frac{1+\sqrt{5}}{2}$, $\frac{-q+\sqrt{p^2+q^2}}{p} > \frac{-1+\sqrt{5}}{2}$, so that, subtracting the second from the first, $2q/p < 1$, and obviously p and q are positive in a recurring period.

This completes our proof.

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* *Vide* p. 91, Perron, *loc. cit.*

5·5 1. We will now point out an application of the last two lemmas to the most rapidly convergent continued fractions. Tietze* has shown that such continued fractions are characterised by the property that the complete quotients are, after a certain point, always greater than $\frac{1+\sqrt{5}}{2}$. The B.c.f.'s are therefore of this class. We have proved elsewhere† that the only transformations (apart from the P-transformation) which convert a simple continued fraction into one of the most rapidly convergent h.r.c.f.'s are the annihilatory transformations which we have called the C_1, C_2, and $C_1C'_2$ types. The effect of an annihilatory transformation applied to a unit partial quotient is obviously to increase the following complete quotient by 1, without affecting the preceding complete quotient.

From these considerations, we see that a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$ will occur in any most rapidly convergent h.r.c.f. development (not involving a P-transformation) of $\sqrt{R}/Q_0$ (> 1, and in the standard form), when and only when either $\frac{q+\sqrt{p^2+q^2}}{p}$ or $\frac{q+p+\sqrt{p^2+q^2}}{p}$ occurs in the simple continued fraction development. But $\frac{q+p+\sqrt{p^2+q^2}}{p}$ is not a reduced surd in Perron's sense‡ and therefore cannot occur in the recurring period of the simple continued fraction, while $\frac{q+\sqrt{p^2+q^2}}{p}$ will occur just once in the recurring period under the conditions of Lemma (3).

Hence, every most rapidly convergent h.r.c.f. development of $\sqrt{R}/Q_0$ (not involving a P-transformation) will contain in its period $\frac{q+p+\sqrt{p^2+q^2}}{p}$ as a complete quotient just once when the unit partial quotient corresponding to $\frac{q+\sqrt{p^2+q^2}}{p}$ in the simple continued fraction is not annihilated.

If $\sqrt{R}/Q_0 = (a_0, a_1, a_2, \cdots a_p, 1_{(l+2)}, a_p, \cdots a_1, 2a_0)$, where ξ_{p+2l+2} is the only complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, the result of applying the C_1-transformation gives the complete quotient $1+\xi_{p+2l+2}$, while the C_2-transformation will annihilate the unit partial quotient corresponding to

* Tietze, H., *Montashefte fü· Mathematik und Physik*, 1913, 24.

† Ayyangar, A. A. K., *Maths. Student*, 1938, 6.

‡ *Vide* p. 79, Perron, *loc. cit.*

ξ_{p+2t+2} and so there will be no complete quotient of the form in question. To preserve the complete quotient, we may also apply the eclectic transformation $C_1 C^e_2$, provided that C_1 process is continued at least until it annihilates the $(2t+1)-$ th central unit partial quotient. Hence we may state that it is possible to have a complete quotient of the form in question in the B.c.f. development as well as in the continued fraction to the nearest integer, but not in the singular continued fraction (all of which do not involve the P-transformation*).

5·6. We are now in a position to resume our original thread of discussion and study the nature of the recurring period of the B.c.f. development of $\frac{\sqrt{R}}{Q_0}$. We at once recognize three possible types:

Type I.—This occurs when the recurring cycle does not contain any complete quotient of the form $(1-g)^{-1}$, *i.e.*, $\frac{q+p+\sqrt{p^2+q^2}}{p}$, g being a special critical fraction pertaining to R. Evidently, this type must occur when R cannot be expressed as the sum of two squares, or when $\sqrt{R}/Q_0$ does not satisfy the conditions of Lemma (3). We will presently show that the characteristic property of this type is that it simulates the simple continued fraction period in its symmetries and also in the property of the last partial quotient. *e.g.*, $\sqrt{46} = 7 - \underset{\times}{\frac{1}{5}} - \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} - \frac{1}{5} - \underset{\times}{\frac{1}{14}}$.

Type II.—This occurs when the recurring cycle contains a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$. We call this 'almost' symmetrical, as the symmetries are slightly disturbed, as for example, $\sqrt{58} = 8 - \underset{\times}{\frac{1}{3}} - \frac{1}{2} - \frac{1}{2} - \underset{\times}{\frac{1}{16}}$.

Type III.—This is an extreme case of Type II, with only two terms in the recurring period, *e.g.*, $\sqrt{n^2+n+\frac{1}{2}} = n+1 - \underset{\times}{\frac{1}{2}} + \underset{\times}{\frac{1}{2n+1}}$.

5·61. Let $\xi_0 = b_0 + \underset{\times}{\frac{\epsilon_1}{b_1}} + \dots \frac{\epsilon_{k-1}}{b_{k-1}} + \underset{\times}{\frac{\epsilon_k}{b_0}}$, $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$,

$$\bar{\xi}_v = \frac{P_v - \sqrt{R}}{Q_v}, \text{ and } \zeta_v = -\frac{\epsilon_{k-v}}{\bar{\xi}_{k-v}}, \ (v = 0, 1, \cdots k-1),$$

where ξ_v is the v-th successor of ξ_0, $\xi_k = \xi_0$, $\bar{\xi}_k = \bar{\xi}_0$.

* *Vide Maths. Student*, **6**, 63; and *Journal of the Mysore University*, Vol. I, Part II, Note (2), Th. II.

Then, as in the simple continued fraction it is easily seen that

$$\zeta_0 = b_{k-1} + \frac{\epsilon_{k-1}}{\dot{b}_{k-2}} + \frac{\epsilon_{k-2}}{b_{k-3}} + \cdots + \frac{\epsilon_1}{b_0} + \frac{\epsilon_k}{\dot{b}_{k-1}};$$

$$\zeta_{v-1} = b_{k-v} + \frac{\epsilon_{k-v}}{\zeta_v} = \frac{P_{k-v+1} + \sqrt{R}}{Q_{k,v}}. \tag{1}$$

By Theorem VIII, ζ_v is the Bhaskara successor of ζ_{v-1} in all cases except when $Q^2_{k-v-1} + \frac{1}{4}Q^2_{k-v-2} = R$, which implies that $\epsilon_{k-v-1} = -1$, $\epsilon_{k-v} = 1$ and ξ_{k-v-1} is of the form $(1-g)^{-1}$, g being a special critical fraction.

When no successor (immediate or remote) of $\sqrt{R}/Q_0 = \sqrt{D}$ (say)

$$= b_0 + \frac{\epsilon_1}{b_1} + \frac{\epsilon_2}{b_2} + \cdots + \frac{\epsilon_{k-1}}{b_{k-1}} + \frac{\epsilon_k}{b_1} + \frac{\epsilon_2}{b_2} + \cdots \tag{2}$$

is of the form in question, we may write

$$\frac{\epsilon_1}{\sqrt{D} - b_0} = b_1 + \frac{\epsilon_2}{\dot{b}_2} + \cdots + \frac{\epsilon_{k-1}}{b_{k-1}} + \frac{\epsilon_k}{\dot{b}_1}.$$

and by (1), $$\epsilon_1\epsilon_k(\sqrt{D} + b_0) = b_{k-1} + \frac{\epsilon_{k-1}}{\dot{b}_{k-2}} + \cdots + \frac{\epsilon_2}{b_1} + \frac{\epsilon_k}{\dot{b}_{k-1}}. \tag{3}$$

Since the r.h.s. is positive $\epsilon_1\epsilon_k = 1$.

Comparing (2) and (3) which are both B.c.f.'s we get $b_{k-1} = 2b_0$ and the symmetries, which may be characterised thus :

$$b_{v-1} = b_{k-v}\ (v = 2, 3, \cdots k-1);$$
$$Q_{v-1} = Q_{k-v}\ (v = 2, 3, \cdots k-1);$$
$$\epsilon_v \ = \epsilon_{k-v}\ (v = 1, 2, \cdots k-1);$$
$$P_v \ = P_{k-v}\ (v = 1, 2, \cdots k-1).$$

When k is even, or the number of terms in the recurring period is odd, two consecutive b's and two consecutive Q's are equal, *viz.*, $b_{\frac{k-2}{2}} = b_{\frac{k}{2}}$, $Q_{\frac{k-2}{2}} = Q_{\frac{k}{2}}$. When k is odd or the number of terms in the recurring period is even, we have two consecutive ϵ's and P's equal, *viz.*, $\epsilon_{\frac{k-1}{2}} = \epsilon_{\frac{k+1}{2}}$; $P_{\frac{k-1}{2}} = P_{\frac{k+1}{2}}$. Conversely, if two consecutive Q's are equal in the recurring cycle, say, $Q_v = Q_{v-1}$, then $\xi_{k-v} = \frac{P_{k-v} + \sqrt{R}}{Q_{k-v}} = \frac{P_v + \sqrt{R}}{Q_{v-1}} = \xi_v$, so that $v = k/2$ and k is even. Similarly for two consecutive P's, $v = \frac{k+1}{2}$ and k is odd.

THEOREM XIV: *If* $\sqrt{R}/Q_0\ (>1)$ *develops as a Type I. B.c.f., R is a non-square positive integer divisible by* Q_0, *and the number of terms in the recurring cycle is odd, then R is either a sum of two squares or a composite number or is equal to* 3.

Let k be the number of terms in the recurring cycle.

Then, $P^2_{\tau+1} + \epsilon_{\tau+1} Q_{\tau+1} Q_\tau = R$, and $Q_\tau = Q_{\tau+1}$, when $\tau = \frac{k-1}{2}$.

If $\epsilon_{\tau+1} = +1$, R is evidently the sum of two squares.

If $\epsilon_{\tau+1} = -1$, and R is a prime, $P_{\tau+1} + Q_{\tau+1} = R$, and $P_{\tau+1} - Q_{\tau+1} = 1$, so that $Q_{\tau+1} = \frac{R-1}{2} < \sqrt{R}$, and therefore R is either 3 or 5. In both these cases, it is easily verified that $k = 1$.

When R is neither 3, nor a sum of two squares, $\epsilon_{\tau+1} = -1$, and R is composite.

Cor.—When R is a prime ($\neq 3$) and is not the sum of two squares, k is even.

5·62. If in the B.c.f. development of $\sqrt{R}/Q_0$ given in (2) of § 5·61, ξ_{k-2} happens to be of the form $(1-g)^{-1}$, then ξ_{k-1} is the conjugate of $-g$ (*vide* Theorem X), and being the predecessor of ξ_k is also of the form $\sqrt{D}+\mu$, where μ is an integer; *i.e.*, $p-q$ is divisible by $2q$ $(p > 2q > 0)$. Hence, we may put $p = (2n+1)q$, $R = p^2+q^2 = q^2(4n^2+4n+2)$,

$\xi_{k-1} = n + \frac{\sqrt{R}}{2q}$, so that $\sqrt{D}$ is of the form $\sqrt{4n^2+4n+2}/2$.

The B.c.f. development of $\frac{\sqrt{4n^2+4n+2}}{2}$ is $n+1-\frac{1}{\underset{\times}{2}}{}_{+}\frac{1}{\underset{\times}{2n+1}}$.

This is what we have called Type III.

5·63. As we have already seen, the recurring period in Type II will contain one and only one complete quotient of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}$, and therefore, the recurring cycle will be merely a cyclic permutation of that of this complete quotient.

By Theorem XIII, $(1-g)^{-1} = \frac{p+q+\sqrt{p^2+q^2}}{p}$, $(p > 2q > 0)$ develops as a pure recurring B.c.f. We will now proceed to study its nature.

Let $\xi'_0 = (1-g)^{-1} = \frac{p+q+\sqrt{R}}{p}$; $\xi'_\tau = \frac{P'_\tau+\sqrt{R}}{Q'_\tau}$; $\xi'_0 = \frac{p+q-\sqrt{R}}{p}$

By Theorem X, $\xi'_0 = 2 + 1/(\text{conjugate of} - g) = 2 + \dfrac{2q}{p - q + \sqrt{R}}$

$$= 2 + \frac{1}{\underset{\times}{b'_1}} + \frac{\epsilon'_2}{b'_2} + \frac{\epsilon'_3}{b'_3} + \cdots + \frac{\epsilon'_{k'-1}}{b'_{k'-1}} + \frac{\epsilon'_{k'}}{\underset{\times}{2}}(-\xi'_{k'}) \quad (1)$$

As in § 5·6 1, we write

$$\epsilon'_{k'}(-1 + \text{conjugate of} - g) = -\frac{\epsilon'_k}{\xi'_0} \cdot b'_{k'-1} + \frac{\epsilon'_{k'-1}}{\underset{\times}{b'_{k'-2}}} + \cdots + \frac{\epsilon'_2}{b'_1} + \frac{1}{2} + \frac{\epsilon'_{k'}}{\underset{\times}{b'_{k'-1}}},$$

which will be a B.c.f. development for the first $(k'-2)$ terms, since ξ'_v is not of the form $(1-g)^{-1}$ for $v = 1, 2, 3, \cdots (k'-1)$.

Hence,

$$\text{conjugate of } -g = -1 - \epsilon'_{k'} b'_{k'-1} - \frac{\epsilon'_{k'} \epsilon'_{k'-1}}{b'_{k'-2}} + \frac{\epsilon'_{k'-2}}{b'_{k'-3}} + \cdots + \frac{\epsilon'_3}{b'_2 + \cdots} \quad (2)$$

From (1) the conjugate of $-g = b'_1 + \dfrac{\epsilon'_2}{b'_2} + \cdots + \dfrac{\epsilon'_{k'}}{2} + \cdots$ (3)

Comparing the first $(k'-1)$ complete quotients and the first $(k'-2)$ terms of (2) and (3), which correspond to the B.c.f. developments of the same number we obtain the following properties of (1):

(i) $-\epsilon'_{k'} b'_{k'-1} = b'_1 + 1$, a positive integer, so that $\epsilon'_{k'} = -1$, and $b'_{k'-1} = b'_1 + 1$.

(ii) The symmetries $b'_v = b'_{k'-v}$ $(v = 2, 3, \cdots k-2)$;

$Q'_v = Q'_{k'-v}$ $(v = 1, 2, \cdots k-1)$;

$\epsilon'_v = \epsilon'_{k'-v+1}$ $(v = 2, 3, \cdots k-1)$;

$P'_v = P_{k'-v+1}$ $(v = 2, 3, \cdots k-1)$.

(iii) $P'_1 = p - q$, $Q'_1 = 2q$, $P'_{k'} = P_0' = p + q$, $P'_{k'-1} = q(2n-1) - p$, $Q'_{k'-1} = 2q$, where $n = b'_{k'-1}$ = the integer just greater than p/q when p is not divisible by q, and $n = p/q$ otherwise.*

As in § 5·6 1, we can prove that two consecutive Q's will be equal only when k' is odd, and that two consecutive P's will be equal only when, k' is even. For example, if $P'_v = P'_{v+1}$,

$$\text{then } \xi'_v = \frac{P'_v + \sqrt{R}}{Q'_v} = \frac{P'_{v+1} + \sqrt{R}}{Q'_v} = \frac{P'_{k'-v} + \sqrt{R}}{Q'_{k'-v}}$$

$$= \xi'_{k'-v},$$

so that $v = k' - v$, or $v = k'/2$, *i.e.*, k' is even.

5·6 4. Reverting to the B.c.f. development of $\sqrt{D}$ $(= \sqrt{R}/Q_0 = \xi_0)$ and following the notation of § 5·6 1, we notice that, if ξ_{k-v+1} $(v > 1)$, is the only

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* *Vide* Theorem XIII.

complete quotient of the form $(1-g)^{-1}$ in the period of $\sqrt{D}$, then $b_{k-1} = 2b_0$, $b_{k-2} = b_1$, $\epsilon_1 = \epsilon_{k-1}$, $P_1 = P_{k-1}$, $Q_1 = Q_{k-2}$ (1)

As observed already, the recurring periods of $\sqrt{D}$ and $\xi_{k-\tau-1}$ of the form $(1-g)^{-1}$

viz., $$\left(\frac{\epsilon_1}{b_1}+\frac{\epsilon_2}{b_2}+\dots\dots\dots+\frac{\epsilon_{k-1}}{b_{k-1}}\right), \qquad (\alpha)$$

and $$\left(\frac{1}{b'_1}+\frac{\epsilon'_2}{b'_2}+\dots\dots+\frac{\epsilon'_{k'-2}}{b'_{k'-2}}+\frac{\epsilon'_{k'-1}}{b'_{k'-1}}+\frac{1}{-2}\right) \qquad (\beta)$$

are cyclic permutations of each other.

Now $\frac{\epsilon_{k-1}}{b_{k-1}}$ cannot occur as the first partial fraction in (β), for it will lead to Type III with b_{k-1} as an odd integer, while b_{k-1} is equal to $2b_0$, which is a contradiction. Again if $\frac{\epsilon_{k-1}}{b_{k-1}}$ is the last partial fraction of (β) it will contradict Theorem X, Cor. (2).

Hence $\frac{\epsilon_{k-1}}{b_{k-1}}$ will occur somewhere in the middle of the period (β), coinciding with $\frac{\epsilon'_\tau}{b'_\tau}$, say; then by (1) $P'_{\tau+1} = P'_\tau$, indicating $\tau = k'/2$, and k' is even $(= k-1)$. The period of Type II, *viz.*, (α) is of the form

$$\sqrt{D} = b_0 + \frac{\epsilon_1}{b_1} + \dots + \frac{\epsilon_{\frac{k-3}{2}}}{b_{\frac{k-3}{2}}} + \frac{1}{-2} + \frac{1}{b_{\frac{k+1}{2}}} + \frac{\epsilon_{\frac{k+3}{2}}}{b_{\frac{k+3}{2}}} + \dots + \frac{\epsilon_{k-1}}{2b_0}$$

having an even number of recurring terms and possessing the same symmetries as Type I with the following exceptions:

$b_{\frac{k-1}{2}} = 2$, $\epsilon_{\frac{k-1}{2}} = -1$, $\epsilon_{\frac{k+1}{2}} = 1$, $b_{\frac{k-3}{2}} = b_{\frac{k+1}{2}} + 1$, $P_{\frac{k-1}{2}} = P_{\frac{k+1}{2}}$; which justify our characterisation of this type as 'almost' symmetric.

It may be useful to telescope the results of this section applicable to the case of $\sqrt{R}$ where R is a non-square positive integer, in the form of a theorem.

THEOREM XV: *The period of the B.c.f. development of* $\sqrt{R}$ *is either a completely symmetrical type simulating the corresponding simple continued fraction, or an almost symmetrical type consisting of an even number of partial quotients, say,* 2ν *with a central set of three unsymmetrical terms of the form*

$$\frac{\epsilon_{\nu-1}}{b_{\nu-1}} \frac{1}{-2} + \frac{1}{b_{\nu-1}-1}$$

Cor.—In the almost symmetrical type of $2r$ terms, Q_r is always greater than 4.

For $\frac{P_r + \sqrt{R}}{Q_r}$ is of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}$, so that $Q_r = p > 2q$.

If $q = 1$, $\sqrt{R} = \sqrt{p^2+1} = p + \frac{1}{\underset{\times}{2p}}$, which is not of Type II. Hence $q \geqslant 2$ and $Q_r > 4$. In fact, when $Q_r = 5$, $q = 2$, $\sqrt{29} = 5 + \frac{1}{\underset{\times}{3}} - \frac{1}{2} + \frac{1}{2} + \frac{1}{\underset{\times}{10}}$ (Type II).

We give below a table of B.c.f.'s equal to the square-roots of non-square integers less than 100.

R	B.c.f.	R	B.c.f.
2	$1 + \frac{1}{\underset{\cdot}{2}}$	23	$5 - \frac{1}{\underset{\cdot}{5}} - \frac{1}{\underset{\cdot}{10}}$
3	$2 - \frac{1}{\underset{\cdot}{4}}$	24	$5 - \frac{1}{\underset{\cdot}{10}}$
5	$2 + \frac{1}{\underset{\cdot}{4}}$	26	$5 + \frac{1}{\underset{\cdot}{10}}$
6	$2 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{4}}$	27	$5 + \frac{1}{\underset{\cdot}{5}} + \frac{1}{\underset{\cdot}{10}}$
7	$3 - \frac{1}{\underset{\cdot}{3}} - \frac{1}{\underset{\cdot}{6}}$	28	$5 + \frac{1}{\underset{\cdot}{3}} + \frac{1}{2} + \frac{1}{3} + \frac{1}{\underset{\cdot}{10}}$
8	$3 - \frac{1}{\underset{\cdot}{6}}$	29	$5 + \frac{1}{\underset{\cdot}{3}} - \frac{1}{2} + \frac{1}{2} + \frac{1}{\underset{\cdot}{10}}$
10	$3 + \frac{1}{\underset{\cdot}{6}}$	30	$5 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{10}}$
11	$3 + \frac{1}{\underset{\cdot}{3}} + \frac{1}{\underset{\cdot}{6}}$	31	$6 - \frac{1}{\underset{\cdot}{2}} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{2} - \frac{1}{\underset{\cdot}{12}}$
12	$3 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{6}}$	32	$6 - \frac{1}{\underset{\cdot}{3}} - \frac{1}{\underset{\cdot}{12}}$
13	$4 - \frac{1}{\underset{\cdot}{2}} + \frac{1}{2} - \frac{1}{\underset{\cdot}{8}}$	33	$6 - \frac{1}{\underset{\cdot}{4}} - \frac{1}{\underset{\cdot}{12}}$
14	$4 - \frac{1}{\underset{\cdot}{4}} - \frac{1}{\underset{\cdot}{8}}$	34	$6 - \frac{1}{\underset{\cdot}{6}} - \frac{1}{\underset{\cdot}{12}}$
15	$4 - \frac{1}{\underset{\cdot}{8}}$	35	$6 - \frac{1}{\underset{\cdot}{12}}$
17	$4 + \frac{1}{\underset{\cdot}{8}}$	37	$6 + \frac{1}{\underset{\cdot}{12}}$
18	$4 + \frac{1}{\underset{\cdot}{4}} + \frac{1}{\underset{\cdot}{8}}$	38	$6 + \frac{1}{\underset{\cdot}{6}} + \frac{1}{\underset{\cdot}{12}}$
19	$4 + \frac{1}{\underset{\cdot}{3}} - \frac{1}{5} - \frac{1}{3} + \frac{1}{\underset{\cdot}{8}}$	39	$6 + \frac{1}{\underset{\cdot}{4}} + \frac{1}{\underset{\cdot}{12}}$
20	$4 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{8}}$	40	$6 + \frac{1}{\underset{\cdot}{3}} + \frac{1}{\underset{\cdot}{12}}$
21	$5 - \frac{1}{\underset{\cdot}{2}} + \frac{1}{2} + \frac{1}{2} - \frac{1}{\underset{\cdot}{10}}$	41	$6 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{12}}$
22	$5 - \frac{1}{\underset{\cdot}{3}} + \frac{1}{4} + \frac{1}{3} - \frac{1}{\underset{\cdot}{10}}$	42	$6 + \frac{1}{\underset{\cdot}{2}} + \frac{1}{\underset{\cdot}{12}}$

R	B.c.f.	R	B.c.f.
43	$7-\frac{1}{\dot{2}}+\frac{1}{4}-\frac{1}{7}-\frac{1}{4}+\frac{1}{2}-\frac{1}{\dot{14}}$	72	$8+\frac{1}{\dot{2}}+\frac{1}{\dot{16}}$
44	$7-\frac{1}{\dot{3}}-\frac{1}{4}-\frac{1}{3}-\frac{1}{\dot{14}}$	73	$9-\frac{1}{\dot{2}}+\frac{1}{5}+\frac{1}{5}+\frac{1}{2}-\frac{1}{\dot{18}}$
45	$7-\frac{1}{\dot{3}}+\frac{1}{2}+\frac{1}{3}-\frac{1}{\dot{14}}$	74	$9-\frac{1}{\dot{2}}+\frac{1}{2}-\frac{1}{\dot{18}}$
46	$7-\frac{1}{\dot{5}}-\frac{1}{2}+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}+\frac{1}{2}-\frac{1}{5}-\frac{1}{\dot{14}}$	75	$9-\frac{1}{\dot{3}}-\frac{1}{\dot{18}}$
47	$7-\frac{1}{7}-\frac{1}{14}$	76	$9-\frac{1}{\dot{3}}+\frac{1}{2}-\frac{1}{6}+\frac{1}{4}+\frac{1}{6}-\frac{1}{2}+\frac{1}{3}-\frac{1}{\dot{18}}$
48	$7-\frac{1}{\dot{14}}$	77	$9-\frac{1}{\dot{4}}+\frac{1}{2}+\frac{1}{4}-\frac{1}{\dot{18}}$
50	$7+\frac{1}{\dot{14}}$	78	$9-\frac{1}{\dot{6}}-\frac{1}{\dot{18}}$
51	$7+\frac{1}{\dot{7}}+\frac{1}{\dot{14}}$	79	$9-\frac{1}{\dot{9}}-\frac{1}{\dot{18}}$
52	$7+\frac{1}{\dot{5}}-\frac{1}{4}-\frac{1}{5}+\frac{1}{\dot{14}}$	80	$9-\frac{1}{\dot{18}}$
53	$7+\frac{1}{\dot{4}}-\frac{1}{2}+\frac{1}{3}+\frac{1}{\dot{14}}$	82	$9+\frac{1}{\dot{18}}$
54	$7+\frac{1}{\dot{3}}-\frac{1}{8}-\frac{1}{3}+\frac{1}{\dot{14}}$	83	$9+\frac{1}{\dot{9}}+\frac{1}{\dot{18}}$
55	$7+\frac{1}{\dot{2}}+\frac{1}{2}+\frac{1}{2}+\frac{1}{\dot{14}}$	84	$9+\frac{1}{\dot{6}}+\frac{1}{\dot{18}}$
56	$7+\frac{1}{\dot{2}}+\frac{1}{\dot{14}}$	85	$9+\frac{1}{\dot{5}}-\frac{1}{2}+\frac{1}{4}+\frac{1}{\dot{18}}$
57	$8-\frac{1}{\dot{2}}+\frac{1}{4}+\frac{1}{2}-\frac{1}{\dot{16}}$	86	$9+\frac{1}{\dot{4}}-\frac{1}{3}-\frac{1}{10}-\frac{1}{3}-\frac{1}{4}-\frac{1}{\dot{18}}$
58	$8-\frac{1}{\dot{3}}-\frac{1}{2}+\frac{1}{2}-\frac{1}{\dot{16}}$	87	$9+\frac{1}{\dot{3}}+\frac{1}{\dot{18}}$
59	$8-\frac{1}{\dot{3}}+\frac{1}{7}+\frac{1}{3}-\frac{1}{\dot{16}}$	88	$9+\frac{1}{\dot{3}}-\frac{1}{3}-\frac{1}{3}+\frac{1}{\dot{18}}$
60	$8-\frac{1}{\dot{4}}-\frac{1}{\dot{16}}$	89	$9+\frac{1}{\dot{2}}+\frac{1}{3}+\frac{1}{3}+\frac{1}{2}+\frac{1}{\dot{18}}$
61	$8-\frac{1}{\dot{5}}+\frac{1}{4}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{\dot{16}}$	90	$9+\frac{1}{\dot{2}}+\frac{1}{\dot{18}}$
62	$8-\frac{1}{\dot{8}}-\frac{1}{\dot{16}}$	91	$10-\frac{1}{\dot{2}}+\frac{1}{6}-\frac{1}{6}+\frac{1}{2}-\frac{1}{\dot{20}}$
63	$8-\frac{1}{\dot{16}}$	92	$10-\frac{1}{\dot{2}}+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{\dot{20}}$
65	$8+\frac{1}{\dot{16}}$	93	$10-\frac{1}{\dot{3}}-\frac{1}{5}+\frac{1}{6}+\frac{1}{5}-\frac{1}{3}-\frac{1}{\dot{20}}$
66	$8+\frac{1}{8}+\frac{1}{16}$	94	$10-\frac{1}{\dot{3}}+\frac{1}{3}+\frac{1}{2}-\frac{1}{7}-\frac{1}{10}-\frac{1}{7}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}-\frac{1}{\dot{20}}$
67	$8+\frac{1}{\dot{5}}+\frac{1}{2}+\frac{1}{2}-\frac{1}{9}-\frac{1}{2}+\frac{1}{2}+\frac{1}{5}+\frac{1}{\dot{16}}$	95	$10-\frac{1}{\dot{4}}-\frac{1}{\dot{20}}$
68	$8+\frac{1}{\dot{4}}+\frac{1}{\dot{16}}$	96	$10-\frac{1}{\dot{5}}-\frac{1}{\dot{20}}$
69	$8+\frac{1}{\dot{3}}+\frac{1}{4}-\frac{1}{6}-\frac{1}{4}+\frac{1}{3}+\frac{1}{\dot{16}}$	97	$10-\frac{1}{\dot{7}}-\frac{1}{3}-\frac{1}{2}+\frac{1}{2}-\frac{1}{7}-\frac{1}{\dot{20}}$
70	$8+\frac{1}{\dot{3}}-\frac{1}{4}-\frac{1}{3}+\frac{1}{\dot{16}}$	98	$10-\frac{1}{\dot{10}}-\frac{1}{\dot{20}}$
71	$8+\frac{1}{\dot{2}}+\frac{1}{3}-\frac{1}{9}-\frac{1}{3}+\frac{1}{2}+\frac{1}{16}$	99	$10-\frac{1}{\dot{20}}$

The basic elements of the theory are now fairly complete, and it should be obvious that the B.c.f. has a complicated individuality of its own, that claims recognition and cannot easily be brushed aside by such remarks as " Bhaskara's method is the same as that rediscovered by Lagrange ". We have only constructed " an arch, wherethro' gleam untravelled and partly travelled regions", such as the character of the acyclic part, the transformations that convert the simple continued fraction into the continued fraction to the nearest square, and the associated quadratic forms. These difficult problems need further investigation.