Review for Exam 1

1 A Bit of Review

1.1 Order of Operations

1.1.1 Example

Simplify the expression $-4^2 + 5 - 4 \cdot 3$.

$$-4^2 + 5 - 4 \cdot 3 = -16 + 5 - 12 = -28 + 5 = \boxed{-23}$$

1.1.2 Example

List the order in which operations are being applies to b.

$$b^3 - 2a$$

1) cubing b

2) subtracting 2a (equivalently adding -2a)

1.2 Square Roots and Principal Square Roots

1.2.1 Example

Simplify.

(a) $\sqrt{16} + 5 = 4 + 5 = \boxed{9}$

(b) $\sqrt{27}\sqrt{3} = \sqrt{81} = \boxed{9}$

1.3 Negation

1.3.1 Example

Which of the following is positive?

(a) $e - 3 \approx 2.71 - 3 = -0.29 < 0$ so negative

(b) $e^2 - 6 \approx 7.39 - 6 = 1.39 > 0$ so positive

2 Solving Equations

2.1 Distance on the Number Line: Absolute Value

2.1.1 Example

Find the value.

(a) $|-e|$ $= e$

(b) $|2 - e^2|$ $= e^2 - 2$ (since $e^2 \approx 7.389056\ldots$)

2.2 Solving Equations with One Variable Type

2.2.1 Example

Verify that $x = 4$ is a solution to $2x - 7 = 5(1 - x) + 4x$.

LHS – left hand side
RHS – right " "

LHS: $2 \cdot 4 - 7 = 8 - 7 = 1$

RHS: $5(1-4) + 4 \cdot 4 = 5(-3) + 16 = -15 + 16 = 1$

$\text{LHS} = 1 = \text{RHS}$ ✓

2.2.2 Example

Solve for s.

$$3\left(\frac{2-s}{8}\right) = \frac{s+6}{12}$$

Multiply both sides by 24.

$3\left(\frac{2-s}{8}\right) \cdot 24 = \left(\frac{s+6}{12}\right) \cdot 24 \longrightarrow 3(2-s)3 = 2(s+6)$

$9(2-s) = 2s + 12 \longrightarrow 18 - 9s = 2s + 12 \longrightarrow 18 - 12 = 2s + 9s$

$6 = 11s \longrightarrow \boxed{s = \frac{6}{11}}$

2.2.3 Example

Solve for r.

$$V = \frac{4}{3}\pi r^3$$

$V = \frac{4}{3}\pi r^3 \longrightarrow \frac{3}{4\pi} \cdot V = r^3 \longrightarrow \boxed{r = \sqrt[3]{\frac{3V}{4\pi}}}$

2.2.4 Example

Solve for x.

$$\frac{x}{x+1} = \frac{2}{x^2+x}$$

(*Hint:* Remember to check for extraneous solutions)

$$\frac{x}{x+1}(x^2+x) = \frac{2}{\cancel{x^2+x}}(\cancel{x^2+x}) \longrightarrow \frac{x}{\cancel{x+1}}\cdot x(\cancel{x+1}) = 2$$

$$x^2 = 2 \longrightarrow \boxed{x = \pm\sqrt{2}}$$

Check: $\frac{\sqrt{2}}{\sqrt{2}+1} = \frac{2}{(\sqrt{2})^2+\sqrt{2}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}(\sqrt{2}+1)}$ ✓

$\frac{-\sqrt{2}}{-\sqrt{2}+1} = \frac{2}{(-\sqrt{2})^2-\sqrt{2}} = \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}(\sqrt{2}-1)}$ ✓

2.2.5 Example

Solve for x, $|3x+1| = 5$.

1) if $3x+1 \geq 0$, then $|3x+1| = 3x+1 = 5 \to 3x = 4 \to \boxed{x = 4/3}$

2) if $3x+1 < 0$, then $|3x+1| = -(3x+1) = 5 \longrightarrow 3x+1 = -5 \to 3x = -6 \to \boxed{x = -2}$

2.2.6 Example

Solve the following quadratic equation $x^2 + 5x = 7$.

$$x^2+5x = 7 \longrightarrow x^2+5x-7 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2-4\cdot1\cdot(-7)}}{2} = \frac{-5\pm\sqrt{25+28}}{2} = \boxed{\frac{-5\pm\sqrt{53}}{2}}$$

2.2.7 Example

Solve for t.

$$\frac{1}{(t+1)^2} - 3 = \frac{2}{t+1}$$

$$\frac{1}{\cancel{(t+1)^2}}\cancel{(t+1)^2} - 3(t+1)^2 = \frac{2}{\cancel{t+1}}(t+1)^{\cancel{2}} \longrightarrow 1-3(t+1)^2 = 2(t+1)$$

$$1-3(t^2+2t+1)-2t-2 = 0 \longrightarrow 1-3t^2-\underline{6t}-3-\underline{2t}-2 = 0$$

$$-3t^2-8t-4 = 0$$

$$t = \frac{8\pm\sqrt{(-8)^2-4(-3)(-4)}}{2(-3)} = \frac{8\pm\sqrt{16}}{-6} = \frac{8\pm4}{-6}$$

$\boxed{t_1 = -2}$ $\boxed{t_2 = \frac{4}{-6} = -\frac{2}{3}}$

2.2.8 Example

Find all real solutions to the equation.

$$6x^3 + x^2 = 6x + 1$$

$$6x^3 - 6x + x^2 - 1 = 0$$

$$6(x^3 - x) + x^2 - 1 = 0$$

$$6x(x^2 - 1) + (x^2 - 1) = 0$$

$(x^2-1)(6x+1) = 0$ then by zero product property

$x^2 - 1 = 0 \rightarrow x^2 = 1$ and $6x + 1 = 0 \rightarrow 6x = -1$

$\boxed{x = 1}$ $\boxed{x = -1}$ $\boxed{x = -1/6}$

2.2.9 Example

Find all real solutions to the equation.

$$3 + \sqrt{x} = x$$

$$\sqrt{x} = x - 3$$

Square both sides:

$$(\sqrt{x})^2 = (x-3)^2$$

$$x = x^2 - 6x + 9$$

$$0 = x^2 - 7x + 9$$

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} = \frac{7 \pm \sqrt{49 - 36}}{2}$$

$$= \boxed{\frac{7 \pm \sqrt{13}}{2}}$$

But now we have to check that x is positive since original equation has $\sqrt{x}$, then x must be positive, and $x - 3$ is positive since we have equality $\sqrt{x} = x - 3$

$x_1 = \frac{7+\sqrt{13}}{2} > 0$, $x_2 = \frac{7-\sqrt{13}}{2}$ so far good.

$x_1 - 3 = \frac{7+\sqrt{13}}{2} - 3 = \frac{7+\sqrt{13}-6}{2}$

$= \frac{1+\sqrt{13}}{2} > 0.$ ✓

$x_2 - 3 = \frac{7-\sqrt{13}}{2} - 3 = \frac{7-\sqrt{13}-6}{2}$

$= \frac{1-\sqrt{13}}{2} \approx \frac{1-3.606}{2} < 0.$ ✗

Hence $x = \frac{7+\sqrt{13}}{2}$ is the only solution.

3 The Cartesian Coordinate System

3.1 Graphs of Equations with Two Variables

3.1.1 Example

Is $(5, 4)$ on the graph of $y = 6x - 8$?

$$4 \stackrel{?}{=} 6 \cdot 5 - 8$$
$$\stackrel{?}{=} 30 - 8$$
$$\stackrel{?}{\neq} 22$$

Thus, $(5,4)$ is not on a graph.

3.1.2 Example

Find the x-intercept and y-intercept of the graph $x = y^2 - 5y - 6$.

x-intercept: $y = 0 \rightarrow x = 0^2 - 5 \cdot 0 - 6 = -6$, $x = -6$

y-intercept: $x = 0 \rightarrow 0 = y^2 - 5y - 6 \rightarrow (y-6)(y+1) = 0$

so by zero product property $y - 6 = 0$, $y = 6$

or $y + 1 = 0 \rightarrow y = -1$

Thus, x-intercept: $\boxed{(-6, 0)}$

y-intercept: $\boxed{(0, -1)}$ and $\boxed{(0, 6)}$

3.2 Distance

3.2.1 Example

Use the Pythagorean Theorem to find the distance between the point $(1, -6)$ and $(4, 1)$.

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-1)^2 + (1-(-6))^2}$$
$$= \sqrt{3^2 + 7^2} = \sqrt{9 + 49} = \boxed{\sqrt{58}}$$

3.3 Equations of Circles

3.3.1 Example

Find an equation for the circle with center $(1, -5)$ and radius 3.

Remember: $(x-h)^2 + (y-k)^2 = r^2$, where (h,k) - center, r - radius

Therefore, $(x-1)^2 + (y-(-5))^2 = 3^2 \longrightarrow \boxed{(x-1)^2 + (y+5)^2 = 9}$

3.3.2 Example

Is the graph of $x^2 + y^2 - 6x + 2y = -9$ a circle? If so, find its center and radius.

$$x^2 - 6x + y^2 + 2y + 9 = 0$$
$$(x^2 - 2\cdot 3x + 9) - 9 + (y^2 + 2y + 1) - 1 + 9 = 0$$
$$(x-3)^2 + (y+1)^2 - 9 - 1 + 9 = 0$$
$$(x-3)^2 + (y+1)^2 = 1 = (1)^2$$

$\boxed{\text{Yes}}$ It's a circle with center $\boxed{(3,-1)}$ and radius $\boxed{1}$

3.4 Midpoints

3.4.1 Example

Find the midpoint of the line segment from $(1, -7)$ to $(6, -1)$.

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{1+6}{2}, \frac{-1-7}{2}\right) = \boxed{\left(7/2, -4\right)}$$

3.5 Steepness, Lines, and Rates of Change

3.5.1 Example

Find the slope of the line that passes through $(-4, 2)$ and $(2, -1)$.

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{-1 - 2}{2 - (-4)} = \frac{-3}{6} = \boxed{-1/2}$$

3.5.2 Example

Find an equation for the line that passes through the point $(5, -3)$ and has a slope $\frac{3}{2}$.

Remember: point-slope equation of the line is $y = m(x-h) + k$, where (h,k) point and m - slope.

So $\boxed{y = 3/2\,(x-5) - 3}$, alternative point-slope formula is $y - k = m(x-h)$, I just brought k to the other side.

4 Systems of Equations

4.1 Solutions to Systems of Equations

4.1.1 Example

Find a solution to the following system of equations.

$$3x - 2y = 2$$
$$5x + y = 4$$

$5x + y = 4 \rightarrow y = 4 - 5x$, then

$3x - 2(4-5x) = 2 \rightarrow 3x - 8 + 10x = 2$

$13x = 10 \rightarrow \boxed{x = 10/13}$

$y = 4 - 5 \cdot \frac{10}{13} = \frac{4 \cdot 13 - 50}{13} = \frac{52-50}{13} = \boxed{\frac{2}{13}}$

Solution: $(10/13, 2/13)$

4.1.2 Example

Find a solution to the following system of equations.

$$x^3 + 4y^2 = 12$$
$$x + y^2 = 3$$

$$\begin{array}{rl} & x^3 + 4y^2 = 12 \\ + & -4x - 4y^2 = -12 \\ \hline & x^3 - 4x = 0 \end{array}$$

$x(x^2-4) = 0$

$x = 0$ or $x^2 = 4 \rightarrow x = \pm 2$.

$x = 0$: $0 + y^2 = 3 \rightarrow y = \pm\sqrt{3}$

$x = 2$: $2 + y^2 = 3 \rightarrow y^2 = 1 \rightarrow y = \pm 1$

$x = -2$: $-2 + y^2 = 3 \rightarrow y^2 = 5 \rightarrow y = \pm\sqrt{5}$

Solutions: $\boxed{(0,\sqrt{3}), (0,-\sqrt{3}), (2,1), (2,-1), (-2,\sqrt{5}), (-2,-\sqrt{5})}$

4.1.3 Example

Find the points of intersection between the graphs of $3x - y = 5$ and $(x-1)^2 + y^2 = 16$.

$3x - y = 5 \rightarrow y = 3x - 5$

$(x-1)^2 + (3x-5)^2 = 16 \rightarrow x^2 - 2x + 1 + 9x^2 - 30x + 25 - 16 = 0$

$10x^2 - 32x + 10 = 0$ $\quad x = \frac{32 \pm \sqrt{32^2 - 4 \cdot 10 \cdot 10}}{2 \cdot 10} = \frac{32 \pm \sqrt{1{,}024 - 400}}{20} = \frac{32 \pm \sqrt{624}}{20}$

$= \frac{32 \pm 4\sqrt{39}}{20} \rightarrow x_1 = \frac{32 + 4\sqrt{39}}{20} = \frac{8+\sqrt{39}}{5}$, $x_2 = \frac{8-\sqrt{39}}{5}$

$y_1 = 3 \cdot \frac{8+\sqrt{39}}{5} - 5 = \frac{24 + 3\sqrt{39} - 25}{5} = \frac{3\sqrt{39} - 1}{5}$

$y_2 = 3 \cdot \frac{8-\sqrt{39}}{5} - 5 = \frac{24 - 3\sqrt{39} - 25}{5} = \frac{-3\sqrt{39} - 25}{5}$

Solution: $\left(\frac{8+\sqrt{39}}{5}, \frac{3\sqrt{39}-1}{5}\right), \left(\frac{8-\sqrt{39}}{5}, \frac{-3\sqrt{39}-25}{5}\right)$

3.5.3 Example

Find an equation for the line that passes through the point $(5, 2)$ and is parallel to the line whose equation is $y - 3 = \frac{7}{4}(x - 7)$.

Remember parallel lines have the same slope

thus our new line has a slope $7/4$, and

passes through the point $(5,2)$. Thus

$$y - 2 = 7/4(x-5),$$

$$\boxed{y = 7/4(x-5) + 2}.$$