Key

Review for Exam 2 - Part II

2 Functions

2.1 The Function Concept

2.1.1 Example

Does the equation $s^3 = 5t - 11$ define t as a function of s?

Yes, just solve for t, such as

$$s^3 = 5t - 11 \rightarrow s^3 + 11 = 5t$$

$$\rightarrow t = \frac{s^3 + 11}{5}.$$

2.2 Function Notation

2.2.1 Example

Let $f(x) = x^3 - 4$. Find the following:

(a) What is $\dfrac{f(2) - f(y+1)}{f(1)}$?

1) $f(2) = 2^3 - 4 = 8 - 4 = 4$; $f(y+1) = (y+1)^3 - 4 = y^3 + 3y^2 + 3y + 1 - 4 = y^3 + 3y^2 + 3y - 3$;
$f(1) = 1^3 - 4 = -3$.

2) $$\frac{f(2) - f(y+1)}{f(1)} = \frac{4 - (y^3 + 3y^2 + 3y - 3)}{-3} = \frac{4 - y^3 - 3y^2 - 3y + 3}{-3} = \boxed{-\frac{7 - y^3 - 3y^2 - 3y}{3}}$$

(b) What is $\dfrac{f(x+h) - f(x)}{h}$?

1) $f(x+h) = (x+h)^3 - 4 = x^3 + 3x^2h + 3xh^2 + h^3 - 4$

2) $$\frac{f(x+h) - f(x)}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4 - (x^3 - 4)}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-4} \cancel{-x^3} \cancel{+4}}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h} = \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \boxed{3x^2 + 3xh + h^2}$$

2.3 Piecewise-Defined Functions

2.3.1 Example

Let

$$f(x) = \begin{cases} x-3 & \text{if } x < -2 \\ x^2+1 & \text{if } -2 \le x < 5 \\ \sqrt{x-3} & \text{if } x > 5 \end{cases}$$

- Find $f(-5)$.

$-5 < -2 \longrightarrow f(-5) = -5-3 = -8$

- Find $f(0)$.

$-2 \le 0 < 5 \longrightarrow f(0) = 0^2+1 = 1$

- Find $f(5)$.

5 is not in the domain of this function, thus $f(5)$ doesn't define.

2.4 The Domain of a Function

2.4.1 Example

Find the domain of the following functions.

- $a(x) = x^2 - 2x + 7$.

No division by 0 or negatives under even roots, thus $(-\infty, +\infty)$.

- $b(x) = \dfrac{x-1}{x}$.

There is x at the bottom, thus $x \neq 0$. Thus, $\mathbb{R} - \{0\}$ or $(-\infty, 0) \cup (0, +\infty)$.

- $c(x) = \sqrt{x-2}$.

We have even root, thus $x - 2 \ge 0$ or $x \ge 2$ or $[2, +\infty)$.

- $d(x) = \dfrac{x}{\sqrt{x-1}}$.

We have even root, thus $x-1 \ge 0$ or $x \ge 1$, but $x \neq 1$ otherwise zero at the bottom. Thus, domain is

$(1, +\infty)$ or $x > 1$.

2.5 Average Rates of Change

2.5.1 Example

Let $f(x) = x^3 - 4x + 3$. Find the average rate of change of $f(x)$ with respect to x as x changes from -2 to 2.

1) Evaluate function at $x=-2$ and $x=2$.

$f(2) = 2^3 - 4\cdot 2 + 3 = 8 - 8 + 3 = 3$; $f(-2) = (-2)^3 - 4(-2) + 3 = -8 + 8 + 3 = 3$

2) Avg. rate of change $= \dfrac{f(2) - f(-2)}{2-(-2)} = \dfrac{3-3}{4} = \boxed{0}$

2.5.2 Example

Let $h(x) = 2x^2 - 1$. Find the average rate of change of $h(x)$ on the interval from x to $x+h$. Assume that $h \neq 0$. **Simplify.**

1) Evaluate function at x and $x+h$.

$h(x) = 2x^2 - 1$ and $h(x+h) = 2(x+h)^2 - 1 = 2(x^2 + 2xh + h^2) - 1$

$= 2x^2 + 4xh + 2h^2 - 1$

2) Avg. rate of change $= \dfrac{\cancel{f}h(x+h) - h(x)}{h} = \dfrac{2x^2 + 4xh + 2h^2 - 1 - (2x^2 - 1)}{h}$

$= \dfrac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-1} \cancel{-2x^2} \cancel{+1}}{h} = \dfrac{4xh + 2h^2}{h} = \dfrac{h(4x+2h)}{h} = \boxed{4x + 2h}$

2.6 Operations on Functions

2.6.1 Example

Let $f(x) = \sqrt{x-2}$ and $g(x) = x^2$.

- Find $(f+g)(6)$.

$(f+g)(6) = f(6) + g(6) = \sqrt{6-2} + 6^2 = \sqrt{4} + 36 = 2 + 36 = \boxed{38}$

- Find $(fg)(x)$.

$(fg)(x) = f(x) \cdot g(x) = \boxed{\sqrt{x-2} \cdot x^2}$

- Find $\left(\frac{f}{g}\right)(x)$ and its domain.

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{\sqrt{x-2}}{x^2}}$; $x-2 \geq 0$ and $x \neq 0$. $\boxed{x \geq 2}$ Domain : $\boxed{[2, +\infty)}$

- Find $f(g(3))$.

$f(g(3)) = f(3^2) = f(9) = \sqrt{9-2} = \boxed{\sqrt{7}}$

- Find $g(f(x))$.

$g(f(x)) = g(\sqrt{x-2}) = (\sqrt{x-2})^2$

- Find $f(g(x))$.

$f(g(x)) = f(x^2) = \sqrt{x^2-2}$

2.7 Graph Transformations

2.7.1 Example

Let $g(x) = x^2$. Write $h(x)$ in terms of $g(x)$ and explain how you would transform the graph of g.

- $h(x) = (x-1)^2 + 3$.

$h(x) = (x-1)^2 + 3 = g(x-1) + 3$

- shift right 1 unit
- shift up 3 units

- $h(x) = 3x^2 - 1$.

$h(x) = 3x^2 - 1 = 3g(x) - 1$

- scale vertically by a factor of 3
- shift down 1 unit

2.8 One-to-one Functions and Inverse Functions

2.8.1 Example

Let $f(x) = \dfrac{x-2}{5}$. Find $f^{-1}(x)$.

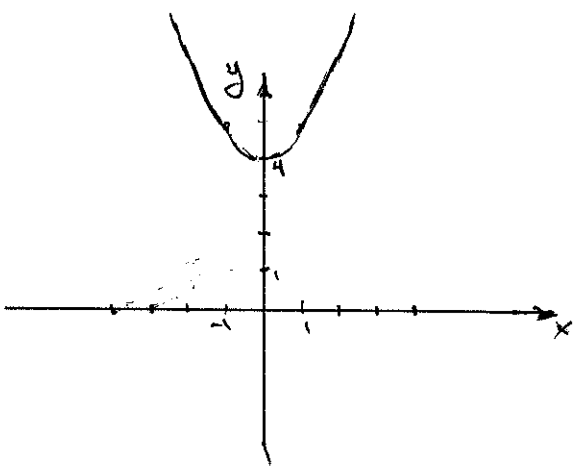
Let $y = f(x)$. Then $y = \dfrac{x-2}{5}$. Switch x and y, so $x = \dfrac{y-2}{5}$.

Solve for y. That's $x = \dfrac{y-2}{5} \longrightarrow 5x = y-2 \longrightarrow y = 5x+2$.

That's $f^{-1}(x) = 5x+2$.

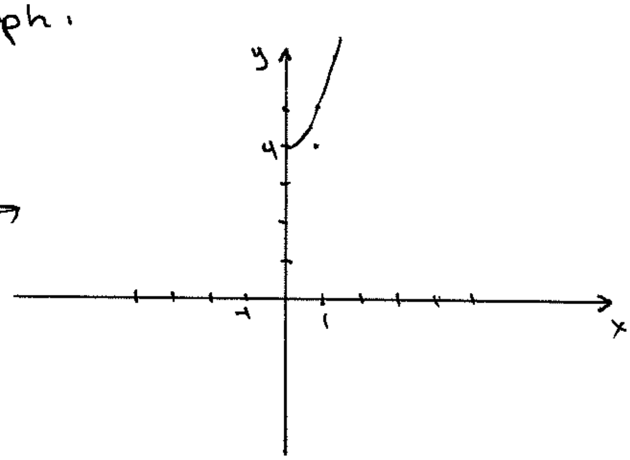
2.8.2 Example Challenging

Let $g(x) = x^2 + 4$. If g has an inverse function, find a formula for $g^{-1}(x)$. If g does not have an inverse function, can you think of a way to restrict the domain of g so that it does have an inverse function. (*Hint:* Restrict the domain of $g(x)$ so that $g(x)$ would become one-to-one function)



doesn't pass the horizontal test

but if we restrict the domain to $x \geq 0$ and get the following graph.



passes the horizontal line test

thus let $g(x) = y$

$y = x^2 + 4$, now replace x and y, so $x = y^2 + 4$ and solve for y.

So $x = y^2 + 4 \longrightarrow y^2 = x - 4 \longrightarrow y = \sqrt{x-4}$.