Review for Final Exam - Part I

1 Exponential and Logarithmic Functions

1.1 Understanding Exponential Functions

1.1.1 Example

A bacteria culture starts out with 200 bacteria and doubles every 5 hours. How many bacteria will there be after 7 hours?

$P(t) = 200a^t$ since $P(0) = 200 \cdot a^0 = 200$.

since it doubles every 5 hours then

$P(5) = 400$. Thus

$400 = 200 \cdot a^5 \longrightarrow 2 = a^5 \longrightarrow a = 2^{1/5}$

$P(t) = 200(2^{1/5})^t = 200 \cdot 2^{t/5}$

$P(7) = 200 \cdot 2^{7/5} \simeq 527.803164$ bacteria.

1.1.2 Example

The half life of some chemical element is 15 days. How much of a 25-gram sample of this element is left after one year?

First, we have to use that $Q(15) = 12.5$ to find the constant a.

$Q(t) = 25a^t$

$Q(15) = 25a^{15}$

$12.5 = 25a^{15}$

$1/2 = a^{15}$

$\left(\frac{1}{2}\right)^{1/15} = a$

$Q(t) = 25\left(\left(\frac{1}{2}\right)^{1/15}\right)^t$

$Q(t) = 25\left(\frac{1}{2}\right)^{t/15}$

$Q(365) = 25\left(\frac{1}{2}\right)^{365/15}$

$\simeq 1.1827 \cdot 10^{-6}$ grams.

1.2 Compound Interest

1.2.1 Example

Suppose you invest \$20,000 in an account that earns 5% interest compounded monthly. How much money will you have in 2 years?

Note that $P_0 = 20{,}000$, $n = 12$, and $r = 0.05$, so

$$P(t) = 20{,}000\left(1 + \frac{0.05}{12}\right)^{12t}$$

$$P(2) = 20{,}000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 2} \approx \$22{,}098.83.$$

1.3 Logarithms

1.3.1 Example

Convert the exponential statement to a logarithmic statement.

(a) $5^4 = 625$

$$\log_5(625) = 4$$

(b) $10^{-2} = \dfrac{1}{100}$

$$\log\left(\frac{1}{100}\right) = -2$$

(c) $e^3 \approx 20.0855369$

$$\ln(20.0855369) \approx 3.$$

1.3.2 Example

Convert the logarithmic statement to an exponential statement.

(a) $\log_4(4^7) = 7$

$$4^7 = 4^7.$$

(b) $\log(100) = 2$

$$10^2 = 100.$$

(c) $\ln(e) = 1$

$$e^1 = e.$$

1.3.3 Example

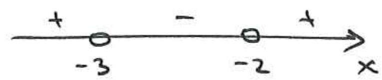
Find the domain of $f(x) = \log(x^2 + 5x + 6)$

$f(x)$ is defined when $x^2 + 5x + 6 > 0$.

$x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3$ or $x = -2$



Thus, the domain of $f(x)$ is $(-\infty, -3) \cup (-2, +\infty)$.

1.3.4 Example

Simplify $e^{2x \ln(3)}$.

$$e^{2x\ln 3} = \left(e^{\ln 3}\right)^{2x} = 3^{2x} = \left(3^2\right)^x = 9^x.$$

1.3.5 Example

Rewrite 3^x as e to a power.

$$3^x = \left(e^{\ln 3}\right)^x = e^{x\ln 3}.$$

1.3.6 Example

Use the properties of logarithms to express $\log\left(\frac{x^2\sqrt{y}}{z^4}\right)$ as a sum and/or difference of there logarithms.

$$\log\left(\frac{x^2\sqrt{y}}{z^4}\right) = \log\left(x^2\sqrt{y}\right) - \log\left(z^4\right)$$

$$= \log(x^2) + \log(\sqrt{y}) - 4\log z$$

$$= 2\log x + \tfrac{1}{2}\log y - 4\log z.$$

1.3.7 Example

Use the properties of logarithms to write the expression using the fewest number of logarithms possible.

$$\ln(x^3+1)+\ln(x)+\ln(z)-3\ln(y)$$

$$\ln(x^3+1)+\ln x+\ln z-3\ln y=\ln((x^3+1)xz)-\ln(y^3)$$

$$=\ln\left(\frac{(x^3+1)x\cdot z}{y^3}\right)$$

1.4 Solving Exponential and Logarithmic Equations

1.4.1 Example

Solve.

$$\ln(x+4)=7$$

By def^n, $\quad x+4=e^7$

$$x=\boxed{e^7-4}$$

1.4.2 Example

Solve. (**Remember to check you answer**)

$x>4 \qquad x>0$

$$\log_7(x-4)+\log_7(x)=2.$$

$$\log_7((x-4)x)=\log_7(7^2)=\log_7 49$$

$$(x-4)x=49$$

$$x^2-4x-49=0$$

$$x=\frac{4\pm\sqrt{16+4\cdot 49}}{2}=\frac{4\pm\sqrt{212}}{2}=\frac{4\pm 2\sqrt{53}}{2}=2\pm\sqrt{53}$$

$$\boxed{x_1=2+\sqrt{53}\simeq 9.28}$$

$x_2=2-\sqrt{53}\simeq -5.28$ ✗

1.4.3 Example

Solve.

$$\frac{3^x + 5}{4} = 3$$

$$3^x + 5 = 12$$

$$3^x = 7$$

$$\log_3(3^x) = \log_3 7$$

$$x = \log_3 7$$

1.4.4 Example

Solve.

$$2^{x-3} = 5^{1-x}$$

$$\log_2(2^{x-3}) = \log_2(5^{1-x})$$

$$x-3 = (1-x)\log_2 5$$

$$x - 3 = \log_2 5 - x\log_2 5$$

$$x + x\log_2 5 = \log_2 5 + 3$$

$$x(1 + \log_2 5) = \log_2 5 + 3 \longrightarrow \boxed{x = \frac{\log_2 5 + 3}{1 + \log_2 5}}$$

1.4.5 Example

A bacteria doubles every 6 hours. How long until the culture triples?

$$P(t) = P_0 a^t$$

$$P(6) = P_0 a^6$$

$$2P_0 = P_0 a^6$$

$$2 = a^6$$

$$a = \sqrt[6]{2} = 2^{1/6}$$

$$P(t) = P_0 (2^{1/6})^t = P_0 2^{t/6}$$

$$3P_0 = P_0 2^{t/6}$$

$$3 = 2^{t/6}$$

$$\log_2 3 = t/6$$

$$t = 6\log_2 3 \approx 9.50978 \text{ hours.}$$