Quiz 9 — 12/01/16

Answer all questions in a clear and concise manner. Answers that are without explanations or are poorly presented may not receive full credit.

Consider the following differential equation with initial condition:

$$\frac{dy}{dx} = \frac{2x}{y}; \qquad y(0) = 2.$$

1. Use Euler's method with step size $h = 1.0$ to approximate $y(3)$ where $y(x)$ is the solution to the initial value problem.

Recall that Euler's Method gives us the formula $y_n = h \cdot f(x_{n-1}, y_{n-1}) + y_{n-1}$. From the given initial conditions we have $x_0 = 0$ and $y_0 = 2$. Some computation gives the following.

i	x_i	y_i	Calculation
0	0	2	
1	1	2	$h \cdot \frac{2x_0}{y_0} + y_0 = 1 \cdot \frac{2 \cdot 0}{2} + 2 = 2$
2	2	3	$h \cdot \frac{2x_1}{y_1} + y_1 = 1 \cdot \frac{2 \cdot 1}{2} + 2 = 3$
3	3	$\frac{13}{3}$	$h \cdot \frac{2x_2}{y_2} + y_2 = 1 \cdot \frac{2 \cdot 2}{3} + 3 = \frac{13}{3}$

Thus $y(3) \approx \frac{13}{3}$.

2. Solve the differential equation and find the actual value of $y(3)$.

We note that $\frac{dy}{dx} = \frac{2x}{y}$ is separable and rewrite it as $y\ dy = 2x\ dx$. Integrating both sides gives $\frac{1}{2}y^2 = x^2 + C$. Plugging in our initial conditions, we find that $C = 2$ and

$$\frac{1}{2}y^2 = x^2 + 2 \implies y^2 = 2x^2 + 4.$$

Solving explicitly for y gives

$$y = \pm\sqrt{2x^2 + 4}.$$

Now we need to decide whether to choose the plus or minus sign. Note that since $\frac{dy}{dx}$ is discontinuous at $x = 0$ we already implicitly chose a range for $y(x)$: either $y > 0$ or $y < 0$. Because the initial condition gives a positive y value, We choose $y > 0$. Hence

$$y(x) = \sqrt{2x^2 + 4}.$$

Now put $x = 3$ to obtain $y(3) = \sqrt{22}$.