HOMEWORK 21 - DIFFERENCE EQUATIONS: STABILITY

1. **(Problem # 3, p. 260)**

Assume the discrete-time population model

$$N_{t+1} = bN_t, \quad t = 0, 1, 2, \ldots$$

Assume also that the population increases by 2% each generation.

(a) Determine b.

(b) Find the size of the population at generation 10 when $N_0 = 20$.

(c) After how many generations will the population size have doubled?

Solution:

(a) Since we assumed that the population increases by 2% each generation this means that the next term in the model is going to be 102% of the last one, but 102% in decimal form if 1.02. Thus, $b = 1.02$.

(b)

$$\begin{aligned}
N_0 &= 20\\
N_1 &= 1.02 \cdot N_0 = 1.02 \cdot 20 = 20.4\\
N_2 &= 1.02 \cdot N_1 = 1.02 \cdot 20.4 = 20.808\\
N_3 &= 1.02 \cdot N_2 = 1.02 \cdot 20.808 = 21.22416\\
N_4 &= 1.02 \cdot N_3 = 1.02 \cdot 21.22416 = 21.6486432\\
N_5 &= 1.02 \cdot N_4 = 1.02 \cdot 21.6486432 = 22.081616064\\
N_6 &= 1.02 \cdot N_5 = 1.02 \cdot 22.081616064 = 22.52324838528\\
N_7 &= 1.02 \cdot N_6 = 1.02 \cdot 22.52324838528 = 22.9737133529856\\
N_8 &= 1.02 \cdot N_7 = 1.02 \cdot 22.9737133529856 = 23.4331876200453\\
N_9 &= 1.02 \cdot N_8 = 1.02 \cdot 23.4331876200453 = 23.9018513724462\\
N_{10} &= 1.02 \cdot N_9 = 1.02 \cdot 23.9018513724462 = 24.3798883998951
\end{aligned}$$

But you can also notice that for example

$$N_2 = 1.02 \cdot N_1 = 1.02 \cdot 1.02 \cdot N_0 = (1.02)^2 N_0$$

and

$$N_3 = 1.02 \cdot N_2 = 1.02 \cdot (1.02)^2 \cdot N_0 = (1.02)^3 N_0,$$

so general formula for N_t is

$$N_t = (1.02)^t N_0 = 20(1.02)^t, \text{ since } N_0 = 20. \tag{1}$$

Thus we can obtain $N_{10} = 20(1.02)^{10} = 24.3798883998951$ which agrees with above computations.

(c) To find out after how many generations will the population size have doubled, we can continuous the above computation and see t after which the population is great than or equal to 40. But the better way to do it is to use the general formula in (1), set it equals to 40 and solve to t. That's

$$40 = 20(1.02)^t \quad\Longleftrightarrow\quad 2 = (1.02)^t \quad\Longleftrightarrow\quad \ln(2) = t\ln(1.02)$$

and

$$t = \frac{\ln(2)}{\ln(1.02)} \approx 35.00279.$$

Thus, after $t = 36$ generations the population size will have doubled, $N_{36} \approx 40.7977$.

Note: We need to round up our t to 36, since $N_{35} \approx 39.9978 < 40$.

2. **(Problem # 9, p. 260)**

Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{2}{3} - \frac{2}{3}x_t^2, \quad t = 0, 1, 2, \ldots$$

Solution:

First, we need to find all the equilibria of the above difference equation. To find those we need to solve the following equation

$$\hat{x} = \frac{2}{3} - \frac{2}{3}\hat{x}^2.$$

Since it's a second degree polynomial we can either use quadratic formula or factoring to obtain to fixed point (i.e. equilibria)

$$\hat{x} = \frac{1}{2} \quad \text{and} \quad \hat{x} = -2.$$

Now, notice that our difference equation has the form $x_{t+1} = f(x_t)$ where $f(x) = \frac{2}{3} - \frac{2}{3}x^2$. Thus using stability criterion we can determine where or not our fixed points are stable.

So $f'(x) = -\frac{4}{3}x$ and

$$\left|f'\left(\frac{1}{2}\right)\right| = \left|-\frac{4}{3}\cdot\frac{1}{2}\right| = \frac{4}{6} < 1 \quad \Longrightarrow \quad \hat{x} = \frac{1}{2} \text{ is locally stable fixed point,}$$

$$\left|f'(-2)\right| = \left|-\frac{4}{3}(-2)\right| = \frac{8}{3} > 1 \quad \Longrightarrow \quad \hat{x} = -2 \text{ is locally unstable fixed point.}$$

3. **(Problem # 13, p. 261)**

(a) Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{5x_t^2}{4+x_t^2}, \quad t = 0, 1, 2 \ldots$$

(b) Use cobwebbing to decide to which value x_t converges as $t \to \infty$ if (i) $x_0 = 0.5$ and (ii) $x_0 = 2$.

Solution:

(a) First, we need to find all the equilibria of the above difference equation. To find those we need to solve the following equation

$$\hat{x} = \frac{5\hat{x}^2}{4+\hat{x}^2}.$$

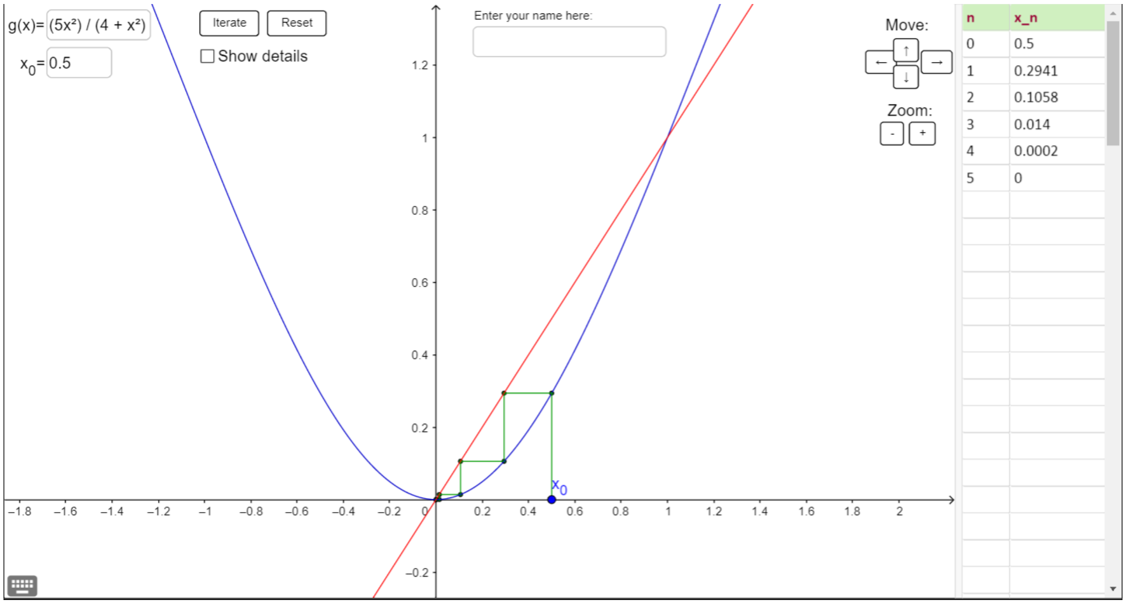
Which is equivalent to $5\hat{x}^2 = \hat{x}(4+\hat{x}^2)$. Since it's a third degree polynomial we can either use polynomial division or graphing calculator to determine that equilibria are

$$\hat{x} = 0, \quad \hat{x} = 1 \quad \text{and} \quad \hat{x} = 4.$$

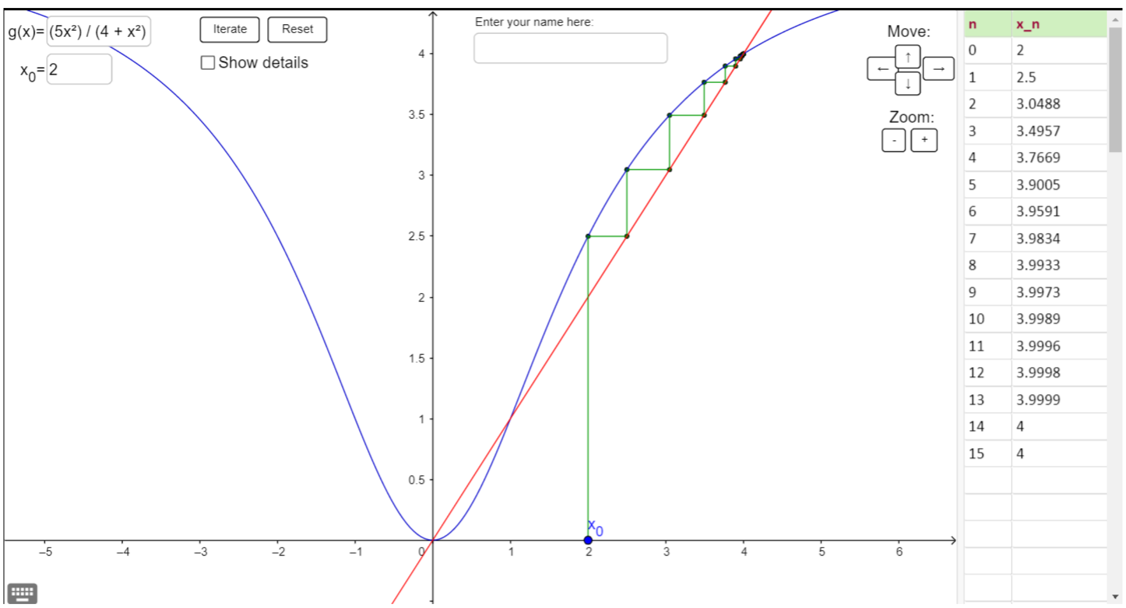
Now, notice that our difference equation has the form $x_{t+1} = f(x_t)$ where $f(x) = \dfrac{5x^2}{4+x^2}$. Thus using stability criterion we can determine where or not our fixed points are stable. So $f'(x) = \dfrac{40x}{(4+x^2)^2}$ and

$$\begin{aligned}
\left|f'(0)\right| &= \left|\frac{40\cdot 0}{(4+0^2)^2}\right| = 0 < 1 &&\Longrightarrow \quad \hat{x} = 0 \text{ is locally stable fixed point,}\\
\left|f'(1)\right| &= \left|\frac{40\cdot 1}{(4+1^2)^2}\right| = \frac{40}{25} = \frac{8}{5} > 1 &&\Longrightarrow \quad \hat{x} = 1 \text{ is locally unstable fixed point,}\\
\left|f'(4)\right| &= \left|\frac{40\cdot 4}{(4+4^2)^2}\right| = \frac{160}{400} = \frac{2}{5} < 1 &&\Longrightarrow \quad \hat{x} = 4 \text{ is locally stable fixed point.}
\end{aligned}$$

(b) For (i) $x_0 = 0.5$ we can see that $x_t \to 0$ as $t \to \infty$.



And for (ii) $x_0 = 2$ we can see that $x_t \to 4$ as $t \to \infty$.



4. **(Problem # 25, p. 261)**

We consider density-dependent population growth models of the form

$$N_{t+1} = R(N_t)N_t.$$

The function $R(N) = e^{r(1-N/K)}$ describes the per capita growth.

Find all nontrivial fixed points $\hat{N}$ (i.e., $\hat{N} > 0$) and determine the stability as a function of the parameter values. We assume that the function parameters are $r > 0$ and $K > 0$.

Solution:

To find all nontrivial fixed point $\hat{N}$, we are going to solve the following equation:

$$\begin{aligned}
\hat{N} &= R(\hat{N})\hat{N} \\
\hat{N} &= e^{r(1-\hat{N}/K)}\hat{N} \\
1 &= e^{r(1-\hat{N}/K)} \quad \text{(since } \hat{N} > 0) \\
\ln(1) &= r(1-\hat{N}/K)\ln(e) \\
0 &= r(1-\hat{N}/K) \\
0 &= 1-\hat{N}/K \quad \text{(since } r > 0) \\
\hat{N} &= K.
\end{aligned}$$

Thus $\boxed{\hat{N} = K}$. And since $K > 0$ by assumption, then we have a nontrivial fixed point.

Now, let's determine conditions under what this fixed point is stable. Since our difference equation has the form $N_{t+1} = f(N_t)$ where $f(x) = e^{r(1-x/K)}x$, then

$$f'(x) = e^{r(1-x/K)}\left(-\frac{r}{K}\right)x + e^{r(1-x/K)}$$

and

$$\left|f'(K)\right| = |-r+1| = |r-1|.$$

Since we are trying to determine condition where fixed point K is stable, then let's set $|f'(K)| < 1$. Thus

$$|r-1| < 1 \quad \text{or} \quad \boxed{0 < r < 2}.$$

Thus, in conclusion, $\hat{N} = K$ is a stable fixed point of the above difference equation is $0 < r < 2$.