Definition

A graded poset $P$ with a minimal and maximal element, denoted $\hat{0}$ and $\hat{1}$ respectively, is *Eulerian* if for all $x < y \in P$ we have

$$\sum_{x \leq z \leq y} (-1)^{rk(z)} = 0$$

Equivalently if $\mu(x, y) = (-1)^{rk(y) - rk(x)}$.

Examples include

- face lattices of polytopes
- face posets of regular CW spheres
- intervals in the Bruhat orders of Coxeter systems
- the lattices of regions of oriented matroids
The uncrossing poset

- rank $\binom{n}{2} + 1$
- $(2n - 1)!! + 1$ elements
- $n$th Catalan number many atoms

**Theorem (Lam)**

*The uncrossing poset is Eulerian.*

**Theorem (Hersh-Kenyon)**

*The uncrossing poset is shellable, moreover it is a CW poset.*
A lattice is a poset $\mathcal{L}$ in which for all $x, y \in \mathcal{L}$ there exists a least upper bound or join $(x \lor y)$ and greatest lower bound or meet $(x \land y)$. In other words the join and meet satisfy:

\[
x \lor y \geq x, y \quad \text{and} \quad z \geq x, y \Rightarrow z \geq x \lor y
\]
\[
x \land y \leq x, y \quad \text{and} \quad z \leq x, y \Rightarrow z \leq x \land y
\]

A join irreducible of a lattice is an element which covers exactly one element.

Let $\text{irr}(\mathcal{L})$ denote the set of join irreducibles of $\mathcal{L}$. 
Definition

A set $X$ is a generating set for a lattice $\mathcal{L}$ if

- $\hat{0}_\mathcal{L} \in X$
- for all $\ell \in \mathcal{L}$, $\ell = x_1 \lor \ldots \lor x_k$ for some $x_i \in X$

Denote the lattice generated by $X$ as $\langle X \rangle$.

Definition

Let $\mathcal{L}$ be a lattice with generating set $\mathrm{gen}(\mathcal{L})$, and $I \subseteq \mathrm{gen}(\mathcal{L}) \setminus \{\hat{0}_\mathcal{L}\}$.

- The contraction of $\mathcal{L}$ by $I$ is $\mathcal{L}/I = \langle j \lor \bigvee_{i \in I} i : j \in \mathrm{gen}(\mathcal{L}) \rangle$.
- The deletion of $\mathcal{L}$ by $I$ is $\mathcal{L}\setminus I = \langle \mathrm{gen}(\mathcal{L}) \setminus I \rangle$.

A minor of $\mathcal{L}$ is a lattice with generating set which is obtained from $\mathcal{L}$ by some sequence of deletions and contractions.

For a minor $M$ denote the generating set by $\mathrm{gen}(M)$.
\[ \mathcal{L} = \langle \hat{0}, 1, 2, 3, 4 \rangle \]

\[ \mathcal{L} \setminus 1 = \langle \hat{0}, 2, 3, 4 \rangle \]

\[ \mathcal{L} / 1 = \langle 1, 12, 13, 1234 \rangle \]

\[ (\mathcal{L} / 1) \setminus (1234) = \langle 1, 12, 13 \rangle \]
Proposition

Let $G$ be a vertex labelled graph and $\mathcal{L}(G)$ its lattice of flats.

Simple vertex labelled minors of $G \leftrightarrow$ minors of $\mathcal{L}(G)$

The bijection is $H \mapsto \mathcal{L}(H)$. 
William Gustafson (U. of Kentucky)  Lattice minors and Eulerian posets  April 13, 2021  11 / 26
The poset of minors

Definition

Let $\mathcal{L}$ be a lattice with generating set. For minors of $\mathcal{L}$ define a partial order by

$$M_1 \leq M_2 \text{ if and only if } M_1 \text{ is a minor of } M_2$$

Let $M_\mathcal{L}$ be the poset of minors of $\mathcal{L}$ together with a minimal element denoted $\emptyset$. 
Some basic results

Proposition

- The atoms of \( M_L \) are the elements of \( L \).
- \( M_L \) is graded by \( \text{rk}(M) = \# \text{gen}(M) \).
- The rank 2 elements of \( M_L \) are minors \( \langle x, x \lor i \rangle \) where \( x \in L \) and \( i \in \text{gen}(L) \) such that \( i \nleq x \).
- \( M_L \) is thin (every rank 2 interval has 4 elements)

Proposition

Let \( L \) be a lattice and \( \text{gen}(L) = \{ \hat{0}_L, \ell_1, \ldots, \ell_n \} \). Let \( \theta : B_n \to L \) be defined by \( \theta(X) = \bigvee_{x \in X} \ell_x \).

The minors of \( L \) are given by \( \langle \theta(X) : X \in \text{irr}(I) \cup \{ \hat{0}_I \} \rangle \) where \( I \) is an interval of \( B_n \).

In particular the minors of \( B_n \) are the intervals.
Some basic results

**Proposition**

Let $\mathcal{L}$ be a lattice with $n + 1$ generators. The interval $[\langle \hat{0} \rangle, \mathcal{L}]$ in $M_\mathcal{L}$ is isomorphic to $B_n$. The subposet of $M_\mathcal{L}$ consisting of contractions of $\mathcal{L}$ is isomorphic to $\mathcal{L}^*$. 

**Proposition**

1. The minor poset $M_{B_n}$ is isomorphic to the face lattice $Q_n$ of the $n$-dimensional cube.

2. Let $C_n$ be the length $n$ chain. The minor poset $M_{C_n}$ is isomorphic to $B_{n+1}$.

$$C_3 = \langle \hat{0}, 1, 2, 3 \rangle \quad C_3 \setminus 1 = \langle \hat{0}, 2, 3 \rangle \quad C_3/1 = \langle 1, 2, 3 \rangle$$

$$
\begin{array}{c}
\hat{0} \\
\downarrow \\
1 \\
\downarrow \\
\hat{0} \\
\end{array} 
\begin{array}{c}
\hat{0} \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\end{array} 
\begin{array}{c}
\hat{0} \\
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2 \\
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3 \\
\end{array}
$$
PL spheres

**Definition**

Given a poset $P$ the *order complex* $\Delta(P)$ is the simplicial complex consisting of all chains in $P$.

A poset $P$ is said to be a *PL sphere* if there is a piecewise linear homeomorphism from $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ to the boundary of a simplex.

**Examples:**

- Face lattices of polytopes
- Intervals of Bruhat orders [Reading]

**Theorem (?)**

*If $P$ is a PL sphere then every interval of $P$ is a PL sphere.*
The zip operation

An element \( z \in P \) is said to be a zipper if

- \( z \) only covers two elements \( x \) and \( y \)
- \( \{ p \in P : p < x \} = \{ p \in P : p < y \} \)
- \( z = x \lor y \)

\( \text{zip}(P, z) \) is the poset obtained from \( P \) by identifying \( x, y \) and \( z \).
The zip operation preserves Eulerianness and PL sphericity.
The main theorem

**Theorem**

Let $\mathcal{L}$ and $\mathcal{K}$ be lattices with generating sets such that there is a join preserving surjection from $\mathcal{L}$ onto $\mathcal{K}$ which descends to a surjection from $\text{gen}(\mathcal{L})$ onto $\text{gen}(\mathcal{K})$.

The poset of minors $M_\mathcal{K}$ can be obtained from $M_\mathcal{L}$ by a sequence of zip operations.

**Corollary**

For any lattice $\mathcal{L}$ with generating set the poset of minors $M_\mathcal{L}$ is Eulerian and a PL sphere.
Proof.

- \( f : \mathcal{L} \to \mathcal{K} \) join preserving,
  \( f(\text{gen}(\mathcal{L})) = \text{gen}(\mathcal{K}) \)
- View \( f \) as a congruence via \( x \equiv y(f) \iff f(x) = f(y) \)
- Define a partial order on edges \( (x_1 < y_1) \leq (x_2 < y_2) \iff x_1 = x_2 \lor z \) and \( y_1 = y_2 \lor z \).
- \( \equiv \) preserves joins if and only if the edges \( x < y \) with \( x \equiv y \) form a lower order ideal
Proof.

- $F : M_L \to M_K$  \( F(\langle X \rangle) = \langle f(X) \rangle \)
- Nontrivial fibers of $F$ are $M, M_x, M_y$ where
  - $x, y \in \text{gen}(M)$
  - $M_y = M \setminus y$
  - $M_x = \begin{cases} M/y & x = \hat{0}_M \\ M \setminus x & x \neq \hat{0}_M \end{cases}$
- Each $M$ is a zipper when these are zipped rank by rank
The cd-index

Let $P$ be a rank $n + 1$ poset with $\hat{0}$ and $\hat{1}$. Let $a$ and $b$ be noncommutative variables.

**Definition**

Let $C$ be a chain in $P$ which contains $\hat{0}$ and $\hat{1}$. Define $w(C) = w_1 \cdot \ldots \cdot w_n$ by

$$w_i(C) = \begin{cases} b & \text{if } C \text{ goes through rank } i \\ (a - b) & \text{if } C \text{ does not go through rank } i \end{cases}$$

**Definition**

The *ab-index* of $P$ is the polynomial

$$\Psi(P) = \sum_{C} w(C)$$

If $\Psi(P)$ is a polynomial in $c = a + b$ and $d = ab + ba$ then this polynomial is the *cd-index* of $P$ (also denoted $\Psi(P)$).
The cd-index continued

**Theorem (Bayer-Billera)**

If $P$ is an Eulerian poset then it has a cd-index.

**Example**

Let $P = B_3$.

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<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$\emptyset &lt; 123$</td>
<td></td>
<td></td>
<td>$(a - b)^2$</td>
</tr>
<tr>
<td>$\emptyset &lt; 1 &lt; 123, \emptyset &lt; 2 &lt; 123, \emptyset &lt; 3 &lt; 123$</td>
<td></td>
<td></td>
<td>$b(a - b)$</td>
</tr>
<tr>
<td>$\emptyset &lt; 12 &lt; 123, \emptyset &lt; 13 &lt; 123, \emptyset &lt; 23 &lt; 123$</td>
<td></td>
<td></td>
<td>$(a - b)b$</td>
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<tr>
<td>$\emptyset &lt; 1 &lt; 12 &lt; 123, \emptyset &lt; 1 &lt; 13 &lt; 123, \emptyset &lt; 2 &lt; 12 &lt; 123$</td>
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<tr>
<td>$\emptyset &lt; 2 &lt; 23 &lt; 123, \emptyset &lt; 3 &lt; 13 &lt; 123, \emptyset &lt; 3 &lt; 23 &lt; 123$</td>
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<td></td>
<td>$b^2$</td>
</tr>
</tbody>
</table>

$$\Psi(B_3) = (a - b)^2 + 3b(a - b) + 3(a - b)b + 6b^2$$

$$= a^2 + 2ba + 2ab + b^2 = c^2 + d$$
The cd-index continued

**Theorem (Karu)**

The cd-index of a Gorenstein* poset has nonnegative coefficients. In particular the cd-index of an Eulerian spherical poset is nonnegative.

**Theorem (Reading)**

Let $z \succ x, y$ be a zipper in $P$, with $z \neq \hat{1}$. If $P$ is Eulerian then

$$\Psi(zip(P, z)) = \Psi(P) - \Psi([\hat{0}, x]) \cdot d \cdot \Psi([z, \hat{1}])$$

**Remark**

If $z = \hat{1}$ then $\Psi(P) = \Psi(zip(P, z)) \cdot c$
A corollary to the main theorem

**Corollary**

Let $\mathcal{L}$ and $\mathcal{K}$ be lattices with generating sets such that there is a join preserving surjection from $\mathcal{L}$ onto $\mathcal{K}$ which descends to a surjection from $\text{gen}(\mathcal{L})$ onto $\text{gen}(\mathcal{K})$.

The following inequality on cd-indices is satisfied coefficientwise.

$$\Psi(M_K) \cdot c^\# \text{gen}(\mathcal{L}) - \# \text{gen}(\mathcal{K}) \leq \Psi(M_\mathcal{L}) \leq \Psi(Q_n)$$

**Example**

Let $\mathcal{L} = \langle \hat{0}, a, b, c \rangle$ as before.

$$\Psi(M_\mathcal{L}) = c^3 + 2cd + 3dc$$

$$\Psi(M_{B_3}) = c^3 + 4cd + 6dc$$