Lattice minors and Eulerian posets

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Definition

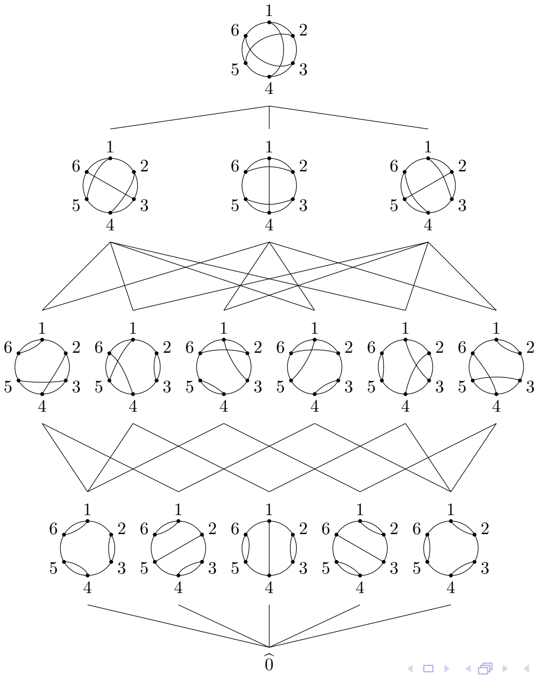
A graded poset P with a minimal and maximal element, denoted $\widehat{0}$ and $\widehat{1}$ respectively, is *Eulerian* if for all $x < y \in P$ we have

$$\sum_{x \leq z \leq y} (-1)^{\mathrm{rk}(z)} = 0$$

Equivalently if $\mu(x, y) = (-1)^{\mathrm{rk}(y) - \mathrm{rk}(x)}$.

Examples include

- face lattices of polytopes
- face posets of regular CW spheres
- intervals in the Bruhat orders of Coxeter systems
- the lattices of regions of oriented matroids


$\widehat{0}$

The uncrossing poset

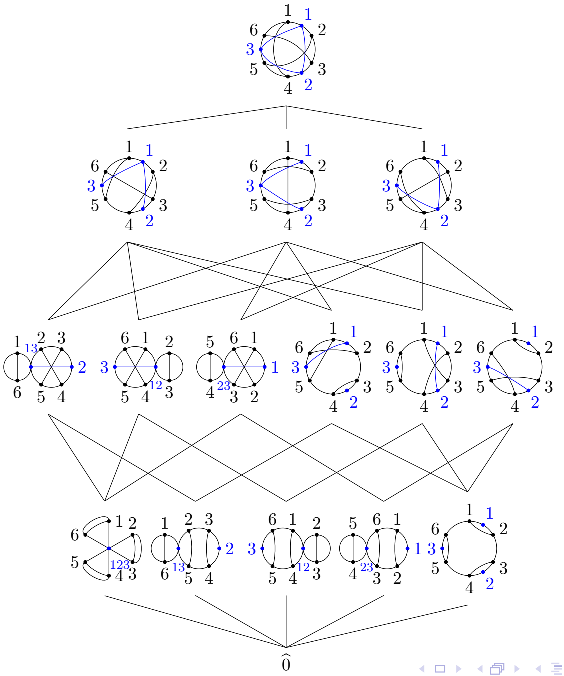
- rank $\binom{n}{2} + 1$
- $(2n-1)!! + 1$ elements
- nth Catalan number many atoms

Theorem (Lam)

The uncrossing poset is Eulerian.

Theorem (Hersh-Kenyon)

The uncrossing poset is shellable, moreover it is a CW poset.



Lattices

Definition

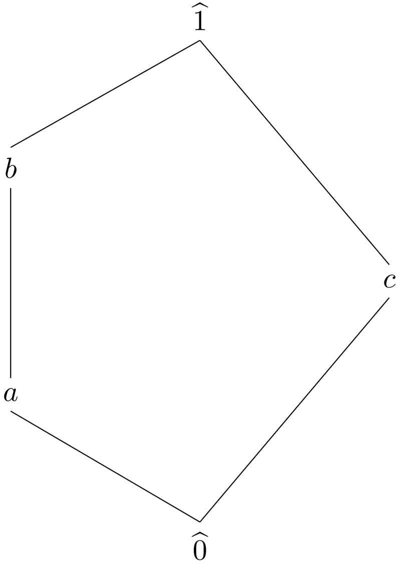
A lattice is a poset $\mathcal{L}$ in which for all $x, y \in \mathcal{L}$ there exists a least upper bound or *join* $(x \vee y)$ and greatest lower bound or *meet* $(x \wedge y)$. In other words the join and meet satisfy:

$$x \vee y \geq x, y \text{ and } z \geq x, y \Rightarrow z \geq x \vee y$$
$$x \wedge y \leq x, y \text{ and } z \leq x, y \Rightarrow z \leq x \wedge y$$

Definition

A *join irreducible* of a lattice is an element which covers exactly one element.

Let $\mathrm{irr}(\mathcal{L})$ denote the set of join irreducibles of $\mathcal{L}$.


$\hat{1}$
b
c
a
$\hat{0}$

Lattice minors

Definition

A set X is a *generating set* for a lattice $\mathcal{L}$ if

- $\widehat{0}_{\mathcal{L}} \in X$
- for all $\ell \in \mathcal{L}$, $\ell = x_1 \vee \ldots \vee x_k$ for some $x_i \in X$

Denote the lattice generated by X as $\langle X \rangle$.

Definition

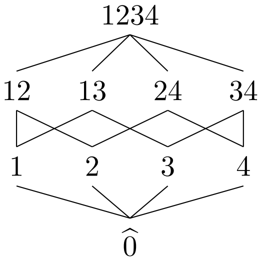
Let $\mathcal{L}$ be a lattice with generating set $\mathrm{gen}(\mathcal{L})$, and $I \subseteq \mathrm{gen}(\mathcal{L}) \setminus \{\widehat{0}_{\mathcal{L}}\}$.

- The *contraction* of $\mathcal{L}$ by I is $\mathcal{L}/I = \langle j \vee \bigvee_{i \in I} i : j \in \mathrm{gen}(\mathcal{L}) \rangle$.
- The *deletion* of $\mathcal{L}$ by I is $\mathcal{L} \setminus I = \langle \mathrm{gen}(\mathcal{L}) \setminus I \rangle$.

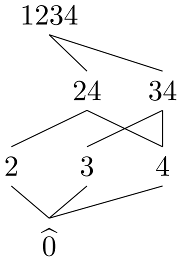
A *minor* of $\mathcal{L}$ is a lattice with generating set which is obtained from $\mathcal{L}$ by some sequence of deletions and contractions.

For a minor M denote the generating set by $\mathrm{gen}(M)$.

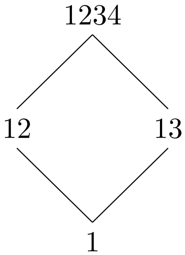
$\mathcal{L} = \langle \widehat{0}, 1, 2, 3, 4 \rangle$



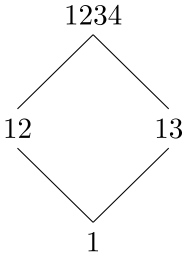
$\mathcal{L} \setminus 1 = \langle \widehat{0}, 2, 3, 4 \rangle$



$\mathcal{L}/1 = \langle 1, 12, 13, 1234 \rangle$



$(\mathcal{L}/1) \setminus (1234) = \langle 1, 12, 13 \rangle$



Graph and matroid minors

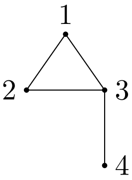
Proposition

Let G be a vertex labelled graph and $\mathcal{L}(G)$ its lattice of flats.

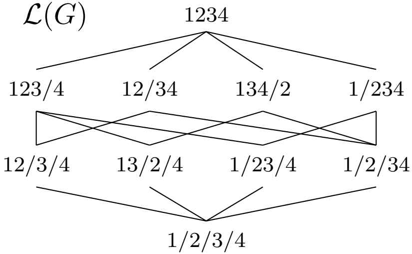
simple vertex labelled minors of $G \leftrightarrow$ minors of $\mathcal{L}(G)$

The bijection is $H \mapsto \mathcal{L}(H)$.

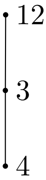
G



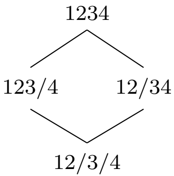
$\mathcal{L}(G)$



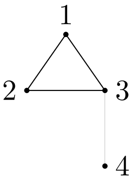
$\widetilde{G/12}$



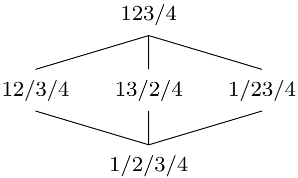
$\mathcal{L}(G/12)$



$G \setminus 34$



$\mathcal{L}(G \setminus 34)$



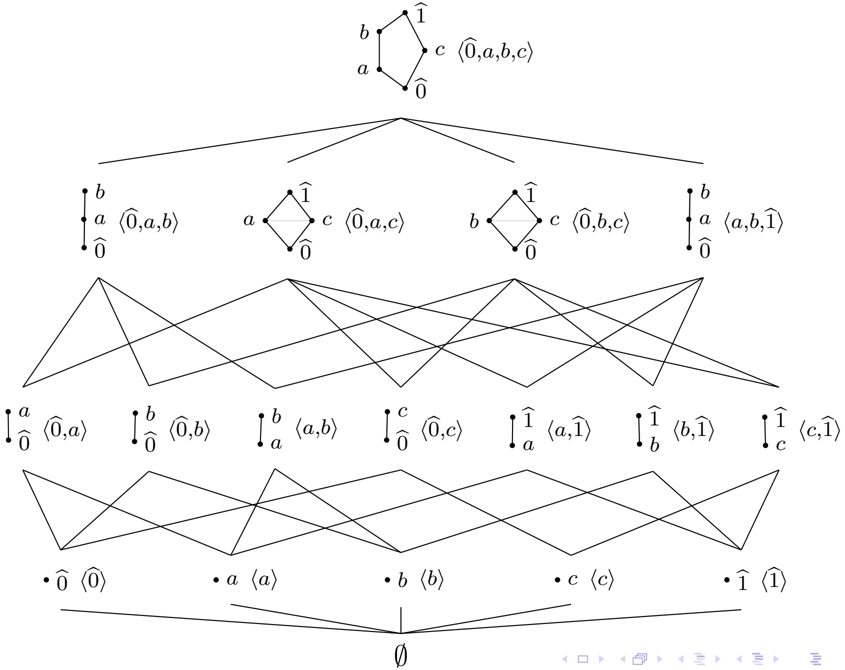
The poset of minors

Definition

Let $\mathcal{L}$ be a lattice with generating set. For minors of $\mathcal{L}$ define a partial order by

$$M_1 \leq M_2 \text{ if and only if } M_1 \text{ is a minor of } M_2$$

Let $M_{\mathcal{L}}$ be the poset of minors of $\mathcal{L}$ together with a minimal element denoted $\emptyset$.


⟨0̂,a,b,c⟩
⟨0̂,a,b⟩
⟨0̂,a,c⟩
⟨0̂,b,c⟩
⟨a,b,1̂⟩
⟨0̂,a⟩
⟨0̂,b⟩
⟨a,b⟩
⟨0̂,c⟩
⟨a,1̂⟩
⟨b,1̂⟩
⟨c,1̂⟩
⟨0̂⟩
⟨a⟩
⟨b⟩
⟨c⟩
⟨1̂⟩
∅

Some basic results

Proposition

- *The atoms of* $M_{\mathcal{L}}$ *are the elements of* $\mathcal{L}$.
- $M_{\mathcal{L}}$ *is graded by* $\operatorname{rk}(M) = \#\operatorname{gen}(M)$.
- *The rank 2 elements of* $M_{\mathcal{L}}$ *are minors* $\langle x, x \vee i \rangle$ *where* $x \in \mathcal{L}$ *and* $i \in \operatorname{gen}(\mathcal{L})$ *such that* $i \not\leq x$.
- $M_{\mathcal{L}}$ *is thin (every rank* 2 *interval has* 4 *elements)*

Proposition

Let $\mathcal{L}$ *be a lattice and* $\operatorname{gen}(\mathcal{L}) = \{\widehat{0}_{\mathcal{L}}, \ell_1, \ldots, \ell_n\}$. *Let* $\theta : B_n \to \mathcal{L}$ *be defined by* $\theta(X) = \bigvee_{x \in X} \ell_x$.

The minors of $\mathcal{L}$ *are given by* $\langle \theta(X) : X \in \operatorname{irr}(I) \cup \{\widehat{0}_I\} \rangle$ *where* I *is an interval of* B_n.

In particular the minors of B_n *are the intervals.*

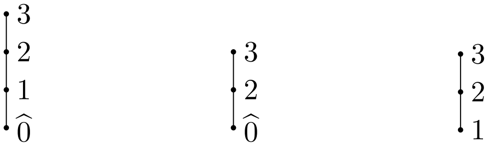
Proposition

Let $\mathcal{L}$ be a lattice with $n+1$ generators. The interval $[\langle\widehat{0}\rangle, \mathcal{L}]$ in $M_{\mathcal{L}}$ is isomorphic to B_n. The subposet of $M_{\mathcal{L}}$ consisting of contractions of $\mathcal{L}$ is isomorphic to $\mathcal{L}^*$.

Proposition

1. *The minor poset M_{B_n} is isomorphic to the face lattice Q_n of the n-dimensional cube.*
2. *Let C_n be the length n chain. The minor poset M_{C_n} is isomorphic to B_{n+1}.*

$C_3 = \langle\widehat{0}, 1, 2, 3\rangle \quad C_3 \setminus 1 = \langle\widehat{0}, 2, 3\rangle \quad C_3/1 = \langle 1, 2, 3\rangle$



PL spheres

Definition

Given a poset P the *order complex* $\Delta(P)$ is the simplicial complex consisting of all chains in P.

A poset P is said to be a *PL sphere* if there is a piecewise linear homeomorphism from $\Delta(P \setminus \{\widehat{0}, \widehat{1}\})$ to the boundary of a simplex.

Examples:

- Face lattices of polytopes
- Intervals of Bruhat orders [Reading]

Theorem (?)

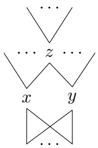
If P is a PL sphere then every interval of P is a PL sphere.

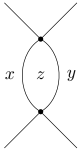
The zip operation

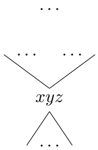
An element $z \in P$ is said to be a *zipper* if

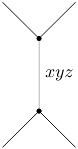
- z only covers two elements x and y
- $\{p \in P : p < x\} = \{p \in P : p < y\}$
- $z = x \vee y$

$\mathrm{zip}(P, z)$ is the poset obtained from P by identifying x, y and z.









The zip operation

Theorem (Reading)

The zip operation preserves Eulerianness and PL sphericity.

The main theorem

Theorem

Let $\mathcal{L}$ and $\mathcal{K}$ be lattices with generating sets such that there is a join preserving surjection from $\mathcal{L}$ onto $\mathcal{K}$ which descends to a surjection from $\operatorname{gen}(\mathcal{L})$ onto $\operatorname{gen}(\mathcal{K})$.

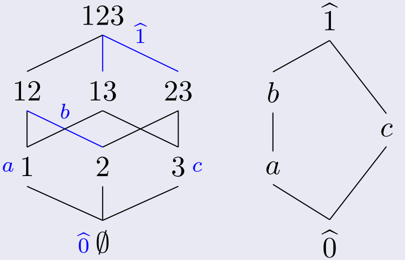
The poset of minors $M_{\mathcal{K}}$ can be obtained from $M_{\mathcal{L}}$ by a sequence of zip operations.

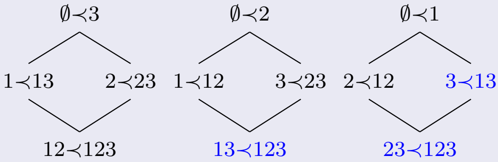
Corollary

For any lattice $\mathcal{L}$ with generating set the poset of minors $M_{\mathcal{L}}$ is Eulerian and a PL sphere.

Proof.

- $f : \mathcal{L} \to \mathcal{K}$ join preserving, $f(\text{gen}(\mathcal{L})) = \text{gen}(\mathcal{K})$
- View f as a congruence via $x \equiv y(f) \Leftrightarrow f(x) = f(y)$
- Define a partial order on edges $(x_1 \prec y_1) \leq (x_2 \prec y_2) \Leftrightarrow x_1 = x_2 \vee z$ and $y_1 = y_2 \vee z$.
- $\equiv$ preserves joins if and only if the edges $x \prec y$ with $x \equiv y$ form a lower order ideal

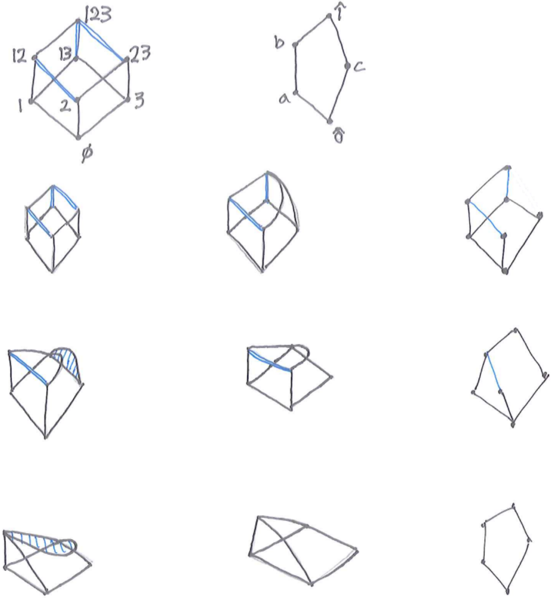




Proof.

- $F : M_{\mathcal{L}} \to M_{\mathcal{K}}$ $\mathrm{F}(\langle X \rangle) = \langle f(X) \rangle$
- Nontrivial fibers of F are M, M_x, M_y where
  - $x, y \in \mathrm{gen}(M)$
  - $M_y = M \setminus y$
  - $M_x = \begin{cases} M/y & x = \widehat{0}_M \\ M \setminus x & x \neq \widehat{0}_M \end{cases}$
- Each M is a zipper when these are zipped rank by rank

□


123
12
13
23
1
2
3
∅
$\hat{1}$
b
c
a
$\hat{0}$

The cd-index

Let P be a rank $n+1$ poset with $\widehat{0}$ and $\widehat{1}$. Let a and b be noncommutative variables.

Definition

Let C be a chain in P which contains $\widehat{0}$ and $\widehat{1}$. Define $w(C) = w_1 \cdot \ldots \cdot w_n$ by

$$w_i(C) = \begin{cases} b & C \text{ goes through rank } i \\ (a-b) & C \text{ does not go through rank } i \end{cases}$$

Definition

The *ab-index* of P is the polynomial

$$\Psi(P) = \sum_C w(C)$$

If $\Psi(P)$ is a polynomial in $c = a+b$ and $d = ab+ba$ then this polynomial is the *cd-index* of P (also denoted $\Psi(P)$).

The cd-index continued

Theorem (Bayer-Billera)

If P is an Eulerian poset then it has a cd-index.

Example

Let $P = B_3$.

$\emptyset < 123$	$(a-b)^2$
$\emptyset < 1 < 123, \emptyset < 2 < 123, \emptyset < 3 < 123$	$b(a-b)$
$\emptyset < 12 < 123, \emptyset < 13 < 123, \emptyset < 23 < 123$	$(a-b)b$
$\emptyset < 1 < 12 < 123, \emptyset < 1 < 13 < 123, \emptyset < 2 < 12 < 123$ $\emptyset < 2 < 23 < 123, \emptyset < 3 < 13 < 123, \emptyset < 3 < 23 < 123$	b^2

$$\Psi(B_3) = (a-b)^2 + 3b(a-b) + 3(a-b)b + 6b^2$$
$$= a^2 + 2ba + 2ab + b^2 = c^2 + d$$

The cd-index continued

Theorem (Karu)

The cd-index of a Gorenstein poset has nonnegative coefficients. In particular the cd-index of an Eulerian spherical poset is nonnegative.*

Theorem (Reading)

Let $z \succ x, y$ be a zipper in P, with $z \neq \hat{1}$. If P is Eulerian then

$$\Psi(\mathrm{zip}(P, z)) = \Psi(P) - \Psi([\hat{0}, x]) \cdot d \cdot \Psi([z, \hat{1}])$$

Remark

If $z = \hat{1}$ then $\Psi(P) = \Psi(\mathrm{zip}(P, z)) \cdot c$

A corollary to the main theorem

Corollary

Let $\mathcal{L}$ and $\mathcal{K}$ be lattices with generating sets such that there is a join preserving surjection from $\mathcal{L}$ onto $\mathcal{K}$ which descends to a surjection from $\operatorname{gen}(\mathcal{L})$ onto $\operatorname{gen}(\mathcal{K})$.

The following inequality on cd-indices is satisfied coefficientwise.

$$\Psi(M_{\mathcal{K}}) \cdot c^{\#\operatorname{gen}(\mathcal{L})-\#\operatorname{gen}(\mathcal{K})} \leq \Psi(M_{\mathcal{L}}) \leq \Psi(Q_n)$$

Example

Let $\mathcal{L} = \langle \widehat{0}, a, b, c \rangle$ as before.

$$\Psi(M_{\mathcal{L}}) = c^3 + 2cd + 3dc$$
$$\Psi(M_{B_3}) = c^3 + 4cd + 6dc$$